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Plate theory and complementary displacement method

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Abstract

With the complementary displacement method introduced in this article regarding plates, it is possible to deduce a constitutive law of a mechanical model from that of the usual three-dimensional (3-D) model. The model studied here is that of a plate undergoing infinitesimal transformations. The method is based on an appropriate kinematic formulation: it decomposes the field of displacements into the sum of a principal displacement, which allows the usual plate theory concepts to be introduced, and a complementary displacement which has been greatly neglected in the classical approach.

The classical hypotheses, whether kinematic like those of Kirchhoff or Reissner, or static like that of plane constraints, are replaced here by the sole hypothesis that the laws of external forces are plate laws: the forces are independent of the complementary displacement and develop no virtual work in a virtual complementary displacement. This being assumed, we show the existence of a global constitutive law linking the fields of generalised deformations and generalised constraints, and also the identical nature of plate equilibrium and three-dimensional solid problems.

A local constitutive law for plates can only be approached provided the plate is thin enough for its vast majority to be far enough from the edge. The most natural way is to assume that the field of generalised deformations is constant; hence the complementary displacement is the solution of an ordinary differential equation which, when solved, gives the constitutive law sought. The cases to which the equation is applied are an elastic isotropic material, producing classical results, and an elastic sandwich plate, leading to new results. Finally, when the field of generalised deformations is polynomial, the fields of complementary displacements and constraints can be found by polynomial identification, thus providing Saint-Venant polynomial solutions, as well as the stiffness matrix of a plate finite element.

1. Introduction

In classical works on Strength of Materials the usual models of beams, plates, shells etc., are presented as approximations of the common model of reference which is the 3-D continuous

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medium. The universal character often attributed to the latter can be questioned, for example when evaluating the behaviour of a cable which has several levels of strands. It remains, nevertheless, indispensable to have at our disposal clear relations between the mechanical properties of each and, more precisely, to be able to deduce the constitutive law of the model chosen from that of the 3-D model.

To achieve this goal, textbooks on Strength of Materials use various hypotheses named after Bernoulli, Kirchhoff, Mindlin or Reissner. All consist of restricting the kinematics by, for example, assuming that the normals to the mean surface remain rectilinear and normal to the mean surface (Kirchhoff), and must be accompanied by static hypotheses, i.e. relative to efforts, e.g. that of plane stresses.

Moreover, most authors do not specify whether the quantities which they neglect, e.g. "transverse shearing", should be regarded as small or null: for the sake of coherence it might be advisable, in the first case to verify after the fact, the supposed smallness, and in the second to provide the corresponding constraint equation and introduce the corresponding unknown constraint effort.

The purpose of this article is to present, in the case of plates submitted to infinitesimal transformations, a general method for establishing links between Strength of Materials models and the 3-D solid. It consists of changing the kinematic representation of the latter by decomposing the total displacement field into the sum of a "principal" displacement and a "complementary" displacement.

The former, which verifies the classical kinematic hypotheses such as Kirchhoff's, is the usual displacement of the new model; it bears the concepts of the new model, e.g. that of the curve of the mean surface and consequently that of the bending moment. The latter, being the difference between the total displacement of the 3-D solid and the "principal" displacement, makes it possible to establish the precise link between the two problems by solving the boundary value problem which naturally arises from the chosen approach.

In fact, the traditional textbook approaches almost ignore the complementary displacement, and it is necessary to turn to journals to find articles which seriously address the 3-D problem, such as those by Koiter (1989), Koiter and Simmonds (1972), Ladeveze (1976, 1980), Levinson (1980), Nair and Reissner (1977), Reissner (1975, 1976, 1985), Touratier (1991), Rychter (1987) or Verchery (1974).

Besides articles of highest quality, as those of the authors quoted above and probably some others, the bibliography of plates is huge and could be counted, if necessary, in cubic meters! Nevertheless, good theoretical papers are not numerous and are concerned with elastic behaviour. On the contrary the present study presents a general approach to plate theory, independent of the tridimensional constitutive equation even if, of course, the application examples given in the last part suppose an elastic behaviour and thus make possible comparisons with classical results.

Two hypotheses are introduced here in order that the 3-D solid problem be replaced by a 2-D problem from which the complementary displacement has disappeared. The first is both kinematic and static and concerns the laws of external efforts: they are independent of the complementary displacement and develop no virtual work in any virtual complementary displacement. This property will subsequently be referred to as $\mathscr{P}$: such an effort law is in fact a plate effort law. In practice, most laws of volume force, such as gravity, include property $\mathscr{P}$; however, this is not the case for surface forces, and particularly for supporting forces.

The two aspects of $\mathscr{P}$ are, as we shall see, closely linked to Saint Venant's principle, according to which it is possible to replace a force law by a related law possessing the property, and it can even be said that the principle consists essentially of expliciting this possibility. According to the hypothesis that all external load laws possess this property, we establish the equivalence between the 3-D problem and a plate problem, to arrive at a "global" constitutive law which links a field of generalised stresses (e.g. bending moment) with a field of generalised strains (e.g. curvature). This is where the boundary value problem for the complementary displacement arises.

The hypothesis that the external load laws possess property $\mathscr{P}$ replaces the usual hypotheses, whether they be static such as that of plane stresses, or kinematic like those of Kirchhoff, Mindlin–Reissner or those which have enriched the kinematics of the latter.

The second hypothesis leads to the definition of a constitutive law applicable to the interior of the plane domain occupied by the plate, and to the calculation of an appropriate constitutive law. It consists once again of referring to Saint Venant's principle in order to replace the boundary value problem by an infinite field problem: as soon as one is sufficiently far from the edge, the internal behaviour is independent of the boundary conditions. Only at this stage does the hypothesis concerning the thinness in relation to the transversal dimensions of the plate intervene implicitly; it is thereby possible to assert that the vast majority of the plate is, in effect, sufficiently far from the edge. Since the famous article by Toupin (1965) on beams, several authors have contributed remarkable works on this question, such as those listed in the bibliography at the end of this article under the names of Knowles (1966), Knowles and Horgan (1969), Horgan (1982), Horgan and Knowles (1983), and Ladeveze (1983).

Nevertheless, no accurate local constitutive law exists for the new model, and we find ourselves confronted with two possibilities: either we seek to create a mechanical model of the plate with its own local constitutive equation or we seek a constitutive equation for a plate finite element.

In the first alternative the obvious approach is to establish the law for a constant field of generalised strains: namely to solve an ordinary differential equation which will give, depending on the generalised strains, the secondary field, followed by that of the stresses, and finally that of the generalised stresses.

In the finite element approach the situation is very different: for a start, the problem of the edge is irrelevant, except if it is to be located at the boundary. Moreover, the field of generalised strains is given in polynomial form depending on the generalised displacements at the nodes; so, in the case of linear elasticity, it is possible to identify exactly the fields of secondary displacements and 3-D stresses in polynomial form, as we shall show, to a degree equal to or less than two, which should normally be feasible to any degree.

These solutions are given, obviously, for an elastic, homogeneous and isotropic material. They are also given for one of the most interesting applications of the theory: the treatment of plates composed of several orthotropic layers. The vital contribution of formal calculation tools (here Mathematica) to the implementation of the method must be underlined here; if publication of the theory, which was conceived in the 70's, was postponed, it was because the tools necessary to implement it were not available at the time. Manual polynomial identification would indeed be a Herculean task.

This article includes only the purely mechanical aspects of the theory, in order to be more easily read by those who have no particular taste for the mathematical notions which are detailed in another article to be published later on where the functional analysis aspects and the questions of duality in coupling the 3-D model and the plate model are dealt with. It will be demonstrated, in particular, that the choice of functional spaces for the latter may be deduced from the choice made for the former, and that it is possible to deduce also two important "closedness" theorems for the plate from the fact that they have already been established for the 3-D solid.

2. Preliminary developments

2.1. *Geometry*

The 3-D Euclidian space is brought to the Cartesian coordinates (x, y, z). The unit vectors of the axes are denoted **i**, **j**, **k**.

In the strict sense of the term, a plate is a solid $\mathscr{S}$ which, in the state of reference, occupies a volume $\mathbf{\Psi}$ of the form:

$$\mathbf{\Psi} = \mathbf{\Omega} \times \left[-\frac{h}{2}, \frac{h}{2} \right]$$

where the thickness h is a constant, $\mathbf{\Omega}$ an open bounded domain of the plane $(x, 0, y)$, with boundary $\partial\mathbf{\Omega}$. Such is the case of the square plate shown in Fig. 1. To enlarge the field of practical application of the theory this sense is extended to solids of variable thickness, symmetrical with regard to plane $(x, 0, y)$:

$$\mathbf{\Psi} = \left\{ (x, y, z)/(x, y) \in \mathbf{\Omega}, |z| \leqslant \frac{h(x, y)}{2} \right\} \tag{1}$$

In this case it is generally assumed that the thickness varies slowly, except possibly in the vicinity of the edge $\partial\mathbf{\Psi}$.

The boundary $\partial\mathbf{\Psi}$ of domain $\mathbf{\Psi}$ is made up of three faces: lower $\partial_1\mathbf{\Psi}$, upper $\partial_2\mathbf{\Psi}$ and side $\partial_3\mathbf{\Psi}$:

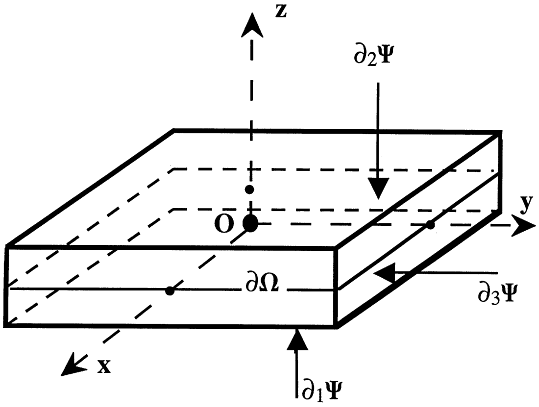

Fig. 1. A plate as a 3-D solid.

$$\partial_1 \boldsymbol{\Psi} = \left\{ (x,y,z)/(x,y) \in \boldsymbol{\Omega}, z = -\frac{h}{2} \right\}$$

$$\partial_2 \boldsymbol{\Psi} = \left\{ (x,y,z)/(x,y) \in \boldsymbol{\Omega}, z = \frac{h}{2} \right\}$$

$$\partial_3 \boldsymbol{\Psi} = \left\{ (x,y,z)/(x,y) \in \partial\boldsymbol{\Omega}, |z| < \frac{h}{2} \right\} \tag{2}$$

The side face can be reduced to a line when the thickness vanishes on the edge. $\bar{\boldsymbol{\Psi}}$ denotes the closed set $\boldsymbol{\Psi} \bigcup \partial\boldsymbol{\Psi}$.

In our first approach no hypothesis is made regarding the thickness, which need not be small in comparison with the other dimensions, nor vary slowly in (x, y). The solid considered here presents no particular features other than its geometrical symmetry with respect to the plane $(x, 0, y)$. It is its kinematic description which will make it a plate.

2.2. *Decomposition of the displacement field and generalised displacements*

The solid $\mathscr{S}$ can be regarded as engendered by the generic particle $\mathscr{P}(x, y, z)$: such is the viewpoint of the 3-D continuous media theory; the elementary mechanical being of the plate theory is not the particle but the "normal", i.e. the material segment:

$$\mathscr{N}(x,y) = \left\{ (x,y,z)/|z| \leqslant \frac{h(x,y)}{2} \right\}$$

The classical hypotheses of plate theories assume that the normal is indeformable, apart from longitudinal dilation or contraction $\varepsilon_{33}(z)$. In the Kirchhoff–Love hypothesis the normal remains normal to the deformed mean plane, while according to the Mindlin–Reissner hypothesis it can lean over the latter. We make neither of these hypotheses, but instead decompose the entire field of displacements $\mathbf{U}$ into:

$$\mathbf{U} = \mathbf{V} + \mathbf{u} \tag{3}$$

where the "principal field of displacements" $\mathbf{V}$ gives the normals a rigid displacement and, moreover, minimises the quadratic error:

$$\|\mathbf{u}\|^2(x,y) = \int_{-(h/2)}^{h/2} |\mathbf{U}(x,y,z) - \mathbf{V}(x,y,z)|^2 \, \mathrm{d}z \tag{4}$$

In other words, when the entire field $\mathbf{U}$ is given, the principal field $\mathbf{V}$ provides the best approximation, of those which keep the normals rigid.

Let us now enter the hypothesis of infinitesimal transformations and determine $\mathbf{V}$ by minimizing (4). As $\mathbf{V}$ keeps the normals indeformable it takes the form:

$$\mathbf{V}(x,y,z) = v_1\mathbf{i} + v_2\mathbf{j} + w\mathbf{k} - \beta_1 z\mathbf{i} - \beta_2 z\mathbf{j}$$

and in column form:

$$\{\mathbf{V}\} = \begin{Bmatrix} v_1 - \beta_1 z \\ v_2 - \beta_2 z \\ w \end{Bmatrix} \tag{5}$$

The quantities v_1, v_2, w, β_1 and β_2 are functions of (x, y) only. Minimising (4) then gives the following (relations):

$$\begin{Bmatrix} v_1 \\ v_2 \\ w \end{Bmatrix} = \frac{1}{h}\int_{-(h/2)}^{h/2} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} \mathrm{d}z; \quad \beta_1 = -\frac{1}{I}\int_{-(h/2)}^{h/2} U_1 z \,\mathrm{d}z; \quad \beta_2 = -\frac{1}{I}\int_{-(h/2)}^{h/2} U_2 z \,\mathrm{d}z \tag{6}$$

where I is the moment of inertia of the normal:

$$I = \frac{h^3}{12}$$

The "complementary field" $\mathbf{u}$, which is the difference between $\mathbf{U}$ and $\mathbf{V}$, is thus, a solution of the following constitutive equations:

$$\int_{-(h/2)}^{h/2} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \mathrm{d}z = 0; \quad \int_{-(h/2)}^{h/2} u_1 z \,\mathrm{d}z = 0; \quad \int_{-(h/2)}^{h/2} u_2 z \,\mathrm{d}z = 0 \tag{7}$$

The five components v_1, v_2, w, β_1 and β_2 taken together, usually grouped in column $\mathbf{q}$, are named "generalised displacement". This is the nodal variable used in many finite element codes. If the vectorial character of these quantities is of particular interest, triplet $(\mathbf{v}, w, \boldsymbol{\beta})$ will be considered bearing in mind that $\mathbf{v}$ and $\boldsymbol{\beta}$ are vectors of plane $(x, 0, y)$, i.e. invariable in plane co-ordinate changes.

It is the particular form (5) of principal field $\mathbf{V}$ which gives rise to the classical concepts of the plate theory, by means of developments which we shall recall in this paragraph.

Notation

Let $\mathbf{I}$ be the identical operator, $\mathbf{A}$ the linear operator defined by the formulae (6) and $\mathbf{A}^+$ the operator defined by formula (5). We have:

$$\mathbf{q} = \mathbf{AU}; \quad \mathbf{V} = \mathbf{A}^+\mathbf{q}; \quad \mathbf{AA}^+ = \mathbf{I} \tag{8}$$

These operators, and their definition domain, are shown in detail in Nayroles.

Remark 1: average displacement and displacement of the mean plane.

The principal field is defined by integrals, the existence of which raises no problem; in particular, $\mathbf{V}(x, y, 0)$ is the average displacement and not the displacement of the mean plane; indeed, one usually has:

$$\mathbf{V}(x, y, 0) \neq \mathbf{U}(x, y, 0)$$

and, moreover, this displacement $\mathbf{U}(x, y, 0)$ may well be undefined almost everywhere on $\mathbf{\Omega}$ while the theory of integration ensures that $\mathbf{V}(x, y, 0)$ is defined whenever $\mathbf{U}$ is summable on $\mathbf{\Psi}$.

Remark 2: elementary plate theory.

In the elementary theory the normals to the mean plane remain orthogonal to its deformed surface. We have just seen that the latter must be replaced by the mean deformed surface denoted by w. The elementary theory is thus the particular case where it is necessary to introduce the constitutive relation:

$$\boldsymbol{\beta} = \mathbf{grad}^{(2)} w \tag{9}$$

n.b.: from now on, as in the above relation where:

$$\mathbf{grad}^{(2)} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}$$

the superscript "(2)" will be attributed to the plane operators of vectorial or tensorial analysis.

Remark 3: abbreviated language.

To avoid repetition the word "field" will be omitted whenever its absence does not lead to confusion. For example, the terms "displacement" and "strain" will be used rather than "displacement field" and "strain field".

2.3. *Virtual work of external forces: generalised efforts*

A solid is submitted to two types of force: on the one hand, "external" forces generally modelled by surface or volume strain densities, or by concentrated forces; and on the other, "internal" forces usually modelled by a field of stress tensors. To introduce the concept of "generalised efforts" in plate modelling, the following vocabulary will be used: the expression "generalised efforts" will be used to refer to the modelling of the external forces only, without it being necessary to specify the adjective "external"; the term "generalised stress" will be synonymous with the expression "generalised internal force", which will not be used.

The external forces will be shown as a volume strain density on $\mathbf{\Psi}$ and as a surface density on its boundary $\partial\mathbf{\Psi}$. Given the particular geometry of the solid, the external forces it is submitted to are expressed as:

$$\mathbf{\Phi} = (\mathbf{f}, \mathbf{F}^1, \mathbf{F}^2, \mathbf{F}^3) \tag{10}$$

where:

- $\mathbf{f}$ is a volume force density exerted on $\mathbf{\Psi}$;
- $\mathbf{F}^1$, $\mathbf{F}^2$, $\mathbf{F}^3$ are surface force densities exerted respectively on $\partial_1\mathbf{\Psi}$, $\partial_2\mathbf{\Psi}$, $\partial_3\mathbf{\Psi}$.

The virtual work of the external forces in virtual displacement $\delta\mathbf{U}$ is:

$$\langle\langle \delta\mathbf{U}, \mathbf{\Phi} \rangle\rangle_3 = \int_{\mathbf{\Psi}} \mathbf{f} \cdot \delta\mathbf{U}\, \mathrm{d}V + \sum_{i=1,2,3} \int_{\partial_i\mathbf{\Psi}} \mathbf{F}^i \cdot \delta\mathbf{U}\, \mathrm{d}\Sigma \tag{11}$$

where $\mathrm{d}V$ is the volume element and $\mathrm{d}\Sigma$ the surface element. The signs $\langle\langle .,.\rangle\rangle_3$ designate the bilinear form of the virtual work in the 3-D model.

The above decomposition is applied to the field $\delta\mathbf{U}$ and the virtual work of the external forces in principal field $\delta\mathbf{V}$ is calculated by using formula (5), resulting in:

$$\langle\langle\delta\mathbf{V},\mathbf{\Phi}\rangle\rangle_3 = \int_\Omega (\mathbf{g}\cdot\delta\mathbf{v}+p\delta w+\mathbf{c}\cdot\delta\boldsymbol{\beta})\,\mathrm{d}x\,\mathrm{d}y+\int_{\partial\Omega}(\mathbf{G}\cdot\delta\mathbf{v}+P\delta w+\mathbf{C}\cdot\delta\boldsymbol{\beta})\,\mathrm{d}s \tag{12}$$

where:

$$\begin{cases} \mathbf{g} = (F_1^1+F_1^2)\mathbf{i}+(F_2^1+F_2^2)\mathbf{j}+\displaystyle\int_{-(h/2)}^{h/2}[f_1\mathbf{i}+f_2\mathbf{j}]\,\mathrm{d}z \\ p = F_3^1+F_3^2+\displaystyle\int_{-(h/2)}^{h/2} f_3\,\mathrm{d}z;\quad \mathbf{c} = -\displaystyle\int_{-(h/2)}^{h/2} z[f_1\mathbf{i}+f_2\mathbf{j}]\,\mathrm{d}z \end{cases} \tag{13}$$

$$\mathbf{G} = \int_{-(h/2)}^{h/2}[F_1^3\mathbf{i}+F_2^3\mathbf{j}]\,\mathrm{d}z;\quad P = \int_{-(h/2)}^{h/2} F_3^3\,\mathrm{d}z;\quad \mathbf{C} = -\int_{-(h/2)}^{h/2} z[F_1^3\mathbf{i}+F_2^3\mathbf{j}]\,\mathrm{d}z \tag{14}$$

- $(\mathbf{g}+p\mathbf{k},\mathbf{c})$ is the "local torsor" in $\mathbf{\Omega}$, i.e., the surface density on $\mathbf{\Omega}$ of the torsor of the forces exerted on the points of an internal normal. Its sum is decomposed into a plane component $\mathbf{g}$ and a component p, while because of the definition of vector $\boldsymbol{\beta}$ which is deduced from the rotation vector of the normal by a rotation of $\pi/2$ in plane $(x,0,y)$, the torque is represented by the vector $\mathbf{c}$ of plane $(x,0,y)$.
- $(\mathbf{G}+P\mathbf{k},\mathbf{C})$ is the "local torsor" in $\partial\mathbf{\Omega}$, i.e., the line density on $\partial\Omega$ of the torsor of the forces exerted on the points of a normal of the side boundary $\partial_3\mathbf{\Psi}$.

Similarly to the way the triplet $\mathbf{q}=(\mathbf{v},w,\boldsymbol{\beta})$ has been named "generalised displacement", the term "generalised effort" will be used for the sextuplet $\mathbf{Q}=(\mathbf{g},p,\mathbf{c},\mathbf{G},P,\mathbf{C})$ associated with $\delta\mathbf{q}$ in the expression (12) of the virtual work of the external forces. This can now be written with the help of the new bilinear form $\langle\langle .,.\rangle\rangle$:

$$\langle\langle\delta\mathbf{q},\mathbf{Q}\rangle\rangle = \int_{\mathbf{\Omega}} (\mathbf{g}\cdot\delta\mathbf{v}+p\delta w+\mathbf{c}\cdot\delta\boldsymbol{\beta})\,\mathrm{d}x\,\mathrm{d}y+\int_{\partial\mathbf{\Omega}}(\mathbf{G}\cdot\delta\mathbf{v}+P\delta w+\mathbf{C}\cdot\delta\boldsymbol{\beta})\,\mathrm{d}s \tag{15}$$

Notation

The linear operator defined by formulae (13) and (14) which associates generalised effort $\mathbf{Q}$ with external force $\mathbf{\Phi}$ is denoted by $\mathbf{A}^{+\mathrm{T}}$:

$$\mathbf{Q} = \mathbf{A}^{+\mathrm{T}}\mathbf{\Phi} \tag{16}$$

Indeed, transposition equality is derived from (12) and (15):

$$\forall\delta\mathbf{q}\forall\mathbf{\Phi}:\quad \langle\langle\delta\mathbf{q},\mathbf{A}^{+\mathrm{T}}\mathbf{\Phi}\rangle\rangle = \langle\langle\mathbf{A}^{+}\delta\mathbf{q},\mathbf{\Phi}\rangle\rangle_3$$

2.4. *Generalised strains*

The total strain is the symmetrical part of the gradient of the total displacement:

$$\boldsymbol{\varepsilon} = \mathbf{grad}_s\mathbf{U} \tag{17}$$

i.e., in index notation:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_j}\right)$$

The decomposition (3) of $\mathbf{U}$ gives rise to the corresponding decomposition of ε into:

$$\boldsymbol{\varepsilon} = \mathbf{e} + \boldsymbol{\eta} \quad \text{with} \quad \mathbf{e} = \mathbf{grad}_s \mathbf{V} \quad \text{and} \quad \boldsymbol{\eta} = \mathbf{grad}_s \mathbf{u} \tag{18}$$

While complementary displacement $\mathbf{u}$ and strain $\boldsymbol{\eta}$ which it generates have not to be developed, principal displacement $\mathbf{V}$ and strain $\mathbf{e}$, on the contrary, convey the classical concepts of plate theory.

By replacing, in (18), $\mathbf{V}$ by its expression (5), we obtain:

$$\mathbf{e} = \begin{bmatrix} \frac{\partial v_1}{\partial x} - z\frac{\partial \beta_2}{\partial x} & \frac{1}{2}\left(\frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial y} - z\frac{\partial \beta_2}{\partial x} - z\frac{\partial \beta_1}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial x} - \beta_1\right) \\ \frac{1}{2}\left(\frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial y} - z\frac{\partial \beta_2}{\partial x} - z\frac{\partial \beta_1}{\partial y}\right) & \frac{\partial v_2}{\partial y} - z\frac{\partial \beta_2}{\partial y} & \frac{1}{2}\left(\frac{\partial w}{\partial y} - \beta_2\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} - \beta_1\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} - \beta_2\right) & 0 \end{bmatrix} \tag{19}$$

which is decomposed, in turn, into the sum of the two terms:

$$\mathbf{e} = \mathbf{e}_{\text{mem}} + \mathbf{e}_{\text{flex}} \tag{20}$$

The first is the strain of the plate in its plane, or "membrane strain"; its matrix takes the form:

$$[\mathbf{e}_{\text{mem}}] = \begin{bmatrix} \frac{\partial v_1}{\partial x} & \frac{1}{2}\left(\frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial y}\right) & 0 \\ \frac{1}{2}\left(\frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial y}\right) & \frac{\partial v_2}{\partial y} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} [\mathbf{e}^{(2)}_{\text{mem}}] & \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \\ \langle 0, 0 \rangle & 0 \end{bmatrix} \tag{21}$$

where $[\mathbf{e}^{(2)}_{\text{mem}}]$ is the plane tensor matrix $\mathbf{e}^{(2)}_{\text{mem}}$. We have:

$$\mathbf{e}^{(2)}_{\text{mem}} = \mathbf{grad}^{(2)}_s \mathbf{v} \tag{22}$$

The second term of (20), $\mathbf{e}_{\text{flex}}$, is interpreted as the strain caused by bending and is written:

$$[\mathbf{e}_{\text{flex}}] = \begin{bmatrix} -z[\chi] & \frac{1}{2}\{\gamma\} \\ \frac{1}{2}\langle \gamma \rangle & 0 \end{bmatrix} \tag{23}$$

with:

$$\{\gamma\} = \begin{Bmatrix} \dfrac{\partial w}{\partial x} - \beta_1 \\ \dfrac{\partial w}{\partial y} - \beta_2 \end{Bmatrix} \quad \text{in other words:} \quad \gamma = \mathbf{grad}^{(2)} w - \boldsymbol{\beta} \tag{24}$$

and with:

$$[\chi] = \begin{bmatrix} \dfrac{\partial \beta_1}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial \beta_2}{\partial x} + \dfrac{\partial \beta_1}{\partial y}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial \beta_2}{\partial x} + \dfrac{\partial \beta_1}{\partial y}\right) & \dfrac{\partial \beta_2}{\partial y} \end{bmatrix} \quad \text{in other words:} \quad \chi = \mathbf{grad}_s^{(2)} \boldsymbol{\beta} \tag{25}$$

Vectorial field $(x, y) \mapsto \boldsymbol{\gamma}(x, y)$ is named "transverse shear strain field".

Quantity $\boldsymbol{\chi}(x, y)$ is a curvature type tensor. The curvature tensor of the deformed mean surface has the matrix:

$$[\mathbf{grad}^{(2)}\,\mathbf{grad}^{(2)}\,w] = \begin{bmatrix} \dfrac{\partial^2 w}{\partial x^2} & \dfrac{\partial^2 w}{\partial x\,\partial y} \\ \dfrac{\partial^2 w}{\partial x\,\partial y} & \dfrac{\partial^2 w}{\partial y^2} \end{bmatrix} = [\chi] + [\mathbf{grad}_s^{(2)} \gamma] \tag{26}$$

The triplet:

$$\boldsymbol{\xi} = (\mathbf{e}^{(2)}_{\text{mem}}, \boldsymbol{\gamma}, \boldsymbol{\chi}) \tag{27}$$

constitutes the "generalised strain" of the plate. It possesses eight independent scalar components; $\mathbf{e}^{(2)}_{\text{mem}}$ and $\boldsymbol{\chi}$ are tensors of plane $(x, 0, y)$ and $\boldsymbol{\gamma}$ a vector of the same.

Formulae (23), (24) and (25) define deformation operator $\mathbf{D}_{\text{p}}$ of the plates; it takes us from generalised displacement $\mathbf{q}$ to generalised strain $\boldsymbol{\xi}$; we write:

$$\boldsymbol{\xi} = \mathbf{D}_{\text{p}}\mathbf{q} \tag{28}$$

and, taking a slight liberty with notation in order to obtain a condensed formula:

$$\boldsymbol{\xi} = \begin{Bmatrix} \mathbf{e}^{(2)}_{\text{mem}} \\ \boldsymbol{\gamma} \\ \boldsymbol{\chi} \end{Bmatrix}; \quad \mathbf{D}_{\text{p}} = \begin{bmatrix} \mathbf{grad}_s^{(2)} & 0 & 0 \\ 0 & \mathbf{grad}^{(2)} & -\mathbf{1} \\ 0 & 0 & \mathbf{grad}_s^{(2)} \end{bmatrix}; \quad \mathbf{q} = \begin{Bmatrix} \mathbf{v} \\ w \\ \boldsymbol{\beta} \end{Bmatrix} \tag{29}$$

It should be noted that it might be useful to add to the second member of (28) a term $\boldsymbol{\xi}_0$ of imposed strain; it could designate strains of thermal origin or those caused by generalised displacements imposed by constraints.

Finally, formula (19) which gives the tensor of the strains due to the principal displacement is written:

$$[\mathbf{e}] = [\mathbf{grad}_{\mathrm{s}}\mathbf{V}] = \begin{bmatrix} [\mathbf{e}_{\mathrm{mem}}^{(2)} - z\boldsymbol{\chi}] & \frac{1}{2}\{\gamma\} \\ \frac{1}{2}\langle\gamma\rangle & 0 \end{bmatrix} \tag{30}$$

where terms $\{\gamma\}$ and $\langle\gamma\rangle$ designate, respectively, the column and line vectors of the components of $\boldsymbol{\gamma}$.

Symbols

Subsequent justification will be given for using the term $\mathbf{B}^{+}$ for the linear operator, defined by (30), which gives the field of principal strains $\mathbf{e}$ generated by the field of generalised strains $\boldsymbol{\xi}$. We can write:

$$\mathbf{e} = \mathbf{B}^{+}\boldsymbol{\xi} = \mathbf{B}^{+}\mathbf{D}_{\mathrm{p}}\mathbf{q} = \mathbf{B}^{+}\mathbf{D}_{\mathrm{p}}\mathbf{A}\mathbf{V} \tag{31}$$

Local integrability of the generalised strains

As in the case of 3-D strains, the problem arises of the local integrability of generalised strains $\boldsymbol{\xi}$. The following proposition can be easily proved:

Proposition 1

Let $\boldsymbol{\xi} = (\mathbf{e}_{\mathrm{mem}}^{(2)}, \boldsymbol{\chi}, \boldsymbol{\gamma})$ be a defined field of generalised strains, twice continuously differentiable in Ω.

There exists, in any Ω simply connex open part, a generalised displacement field $\mathbf{q} = (\mathbf{v}, w, \boldsymbol{\beta})$ such that:

$$\boldsymbol{\xi} = \mathbf{D}_{\mathrm{p}}\mathbf{q}$$

if and only if $\boldsymbol{\xi}$ holds the following conditions of local integrability:

$$\frac{\partial^2 e_{\mathrm{mem}_{22}}^{(2)}}{\partial x^2} + \frac{\partial^2 e_{\mathrm{mem}_{11}}^{(2)}}{\partial y^2} - 2\frac{\partial^2 e_{\mathrm{mem}_{12}}^{(2)}}{\partial x\,\partial y} = 0 \tag{32}$$

$$\frac{\partial^2 \gamma_1}{\partial y\,\partial x} - \frac{\partial^2 \gamma_2}{\partial x^2} = 2\left(\frac{\partial \chi_{12}}{\partial x} - \frac{\partial \chi_{11}}{\partial y}\right)$$

$$\frac{\partial^2 \gamma_1}{\partial y^2} - \frac{\partial^2 \gamma_2}{\partial x\,\partial y} = 2\left(\frac{\partial \chi_{22}}{\partial x} - \frac{\partial \chi_{12}}{\partial y}\right) \tag{33}$$

$$\frac{\partial^2 \chi_{22}}{\partial x^2} + \frac{\partial^2 \chi_{11}}{\partial y^2} - 2\frac{\partial^2 \chi_{12}}{\partial x\,\partial y} = 0 \tag{34}$$

Thus, $\mathbf{v}$ is determined to within one rigid plane displacement, $\boldsymbol{\beta}$ to within one constant additive vector, w to within one constant additive. In other words, $\mathbf{q}$ is defined to within one solid displacement.

A condition of global integrability of the generalised strains will be given later.

2.5. *Virtual work of the internal forces and generalised stresses*

In the 3-D model efforts inside the solid are represented by fields of stress tensors $\boldsymbol{\sigma}$ and their virtual work in field $\delta\boldsymbol{\varepsilon}$ of virtual strain tensors by the integral:

$$\langle \delta\boldsymbol{\varepsilon}, \boldsymbol{\sigma} \rangle_3 = -\int_{\boldsymbol{\Psi}} \boldsymbol{\sigma}: \quad \delta\boldsymbol{\varepsilon}\, \mathrm{d}V \tag{35}$$

and this definition can be extended to tensor fields $\delta\boldsymbol{\varepsilon}$, whether or not they be of form $\mathbf{grad}_s \delta\mathbf{v}$.

Relation (30) defines operator $\mathbf{B}^+$ which, to the value of generalised strain $\boldsymbol{\xi}$ at point (x, y) of Ω associates a tensor $\mathbf{e} = \mathbf{B}^+\boldsymbol{\xi}$ at all points (x, y, z) of the normal. Whether or not field $\delta\boldsymbol{\xi}$ be of the form $\mathbf{D}_p\mathbf{q}$ we can therefore define the virtual work of the internal forces as quantity $\langle \mathbf{B}^+\delta\boldsymbol{\xi}, \boldsymbol{\sigma} \rangle_3$. By first integrating in z we obtain:

$$\langle \mathbf{B}^+\delta\boldsymbol{\xi}, \boldsymbol{\sigma} \rangle_3 = -\int_{\boldsymbol{\Omega}} (\delta\mathbf{e}^{(2)}_{\mathrm{mem}}: \mathbf{N} + \delta\boldsymbol{\beta} \cdot \mathbf{T} + \delta\boldsymbol{\chi}: \mathbf{M})\, \mathrm{d}x\, \mathrm{d}y \tag{36}$$

where quantities $\mathbf{N}$, $\mathbf{T}$, $\mathbf{M}$ are linked to the 3-D stresses by:

$$\mathbf{N} = \begin{bmatrix} \int_{-(h/2)}^{h/2} \sigma_{11}\, \mathrm{d}z & \int_{-(h/2)}^{h/2} \sigma_{12}\, \mathrm{d}z \\ \int_{-(h/2)}^{h/2} \sigma_{12}\, \mathrm{d}z & \int_{-(h/2)}^{h/2} \sigma_{22}\, \mathrm{d}z \end{bmatrix}; \quad \mathbf{T} = \begin{Bmatrix} \int_{-(h/2)}^{h/2} \sigma_{13}\, \mathrm{d}z \\ \int_{-(h/2)}^{h/2} \sigma_{23}\, \mathrm{d}z \end{Bmatrix};$$

$$\mathbf{M} = \begin{bmatrix} \int_{-(h/2)}^{h/2} z\sigma_{11}\, \mathrm{d}z & -\int_{-(h/2)}^{h/2} z\sigma_{12}\, \mathrm{d}z \\ -\int_{-(h/2)}^{h/2} z\sigma_{12}\, \mathrm{d}z & -\int_{-(h/2)}^{h/2} z\sigma_{22}\, \mathrm{d}z \end{bmatrix} \tag{37}$$

$\mathbf{N}$ being the membrane stress tensor, $\mathbf{T}$ the shear effort vector, $\mathbf{M}$ the flexion-torsion tensor. Each of these elements can be named generalised stress, as can the triplet $\mathbf{Z} = (\mathbf{N}, \mathbf{T}, \mathbf{M})$.

Symbols

Let $\mathbf{B}^{+\mathrm{T}}$ be the linear operator defined by formulae (37):

$$\mathbf{Z} = \mathbf{B}^{+\mathrm{T}}\boldsymbol{\sigma} \tag{38}$$

The integral which constitutes the second member of (36) will be:

$$\langle \delta\boldsymbol{\xi}, \mathbf{Z} \rangle = -\int_{\boldsymbol{\Omega}} (\delta\mathbf{e}^{(2)}_{\mathrm{mem}}: \mathbf{N} + \delta\boldsymbol{\beta} \cdot \mathbf{T} + \delta\boldsymbol{\chi}: \mathbf{M})\, \mathrm{d}x\, \mathrm{d}y \tag{39}$$

It is the virtual work of generalised stress $\mathbf{Z}$ in the virtual generalised strain $\delta\boldsymbol{\xi}$. Notation $\mathbf{B}^{+\mathrm{T}}$ is justified by the next equation directly derived from (36), (38) and (39):

$$\forall\delta\xi\forall\boldsymbol{\sigma}:\quad \langle\delta\xi,\mathbf{B}^{+\mathrm{T}}\boldsymbol{\sigma}\rangle=\langle\mathbf{B}^{+}\delta\xi,\boldsymbol{\sigma}\rangle_3$$

2.6. *Plate equilibrium equations*

The equilibrium equation between the sum $\mathbf{\Phi}$ of the external forces and the stress $\boldsymbol{\sigma}$ is that of the virtual work:

$$\forall\delta\mathbf{U}:\quad \langle\langle\delta\mathbf{U},\mathbf{\Phi}\rangle\rangle_3+\langle\mathbf{grad}_s\delta\mathbf{U},\boldsymbol{\sigma}\rangle_3=0 \tag{40}$$

where, for the sake of simplicity, test fields $\delta\mathbf{U}$ are indefinitely differentiables.

Similarly, in this preliminary paragraph, constraint conditions will be regarded as particular external force laws; thus, test fields $\delta\mathbf{U}$ are free of any constraints.

The decomposition of $\delta\mathbf{U}$ into a principal field $\delta\mathbf{V}$ and a complementary field $\delta\mathbf{u}$ enables us to write the following system, clearly equivalent to (40):

$$\forall\delta\mathbf{V}:\quad \langle\langle\delta\mathbf{V},\mathbf{\Phi}\rangle\rangle_3+\langle\mathbf{grad}_s\delta\mathbf{V},\boldsymbol{\sigma}\rangle_3=0 \tag{41.1}$$

$$\forall\delta\mathbf{u}:\quad \langle\langle\delta\mathbf{u},\mathbf{\Phi}\rangle\rangle_3+\langle\mathbf{grad}_s\delta\mathbf{u},\boldsymbol{\sigma}\rangle_3=0 \tag{41.2}$$

where test fields $\delta\mathbf{V}$ are of principal type, i.e., of the form $\mathbf{A}^{+}\delta\mathbf{q}$, while test fields $\delta\mathbf{u}$ are of complementary type, i.e., solutions of constraint equations (7).

Thanks to the above developments it is possible to replace in (41.1), $\delta\mathbf{V}$, $\mathbf{grad}_s\delta\mathbf{V}$, $\mathbf{\Phi}$ and $\boldsymbol{\sigma}$, respectively, by the "generalised" quantities $\delta\mathbf{q}$, $\mathbf{D}_p\delta\mathbf{q}$, $\mathbf{Q}$ and $\mathbf{Z}$ which they produce. The following result is, thus, easily obtained:

Proposition 2

Equation (41.1) is equivalent to equation:

$$\forall\delta\mathbf{q}:\quad \langle\langle\delta\mathbf{q},\mathbf{Q}\rangle\rangle+\langle\mathbf{D}_p\delta\mathbf{q},\mathbf{Z}\rangle=0 \tag{42}$$

where: $\mathbf{Q}=\mathbf{A}^{+\mathrm{T}}\mathbf{\Phi}$ is the generalised effort produced by $\mathbf{\Phi}$ according to series (13) and (14), $\mathbf{Z}=\mathbf{B}^{+}\boldsymbol{\sigma}$ is the generalised effort produced by $\boldsymbol{\sigma}$ according to series (37).

It is then a classical exercise of variation calculation to obtain the following proposition:

Proposition 3

If $(\mathbf{Q},\mathbf{Z})$ satisfies (42) and if: $\mathbf{Q}$ is continuously derivable in $\mathbf{\Omega}$, $\mathbf{Z}$ is composed of continuous fields (some in $\mathbf{\Omega}$, others on $\partial\mathbf{\Omega}$), then $(\mathbf{Q},\mathbf{Z})$ is solution of the following equations:

in $\mathbf{\Omega}$ on $\partial\mathbf{\Omega}$Membrane equations $\mathbf{div}^{(2)}\mathbf{N}=-\mathbf{g}\quad \mathbf{N}\cdot\mathbf{n}=\mathbf{G}$ (43)

Flexion equations $\mathrm{div}^{(2)}\mathbf{T}=-p\quad \mathbf{T}\cdot\mathbf{n}=P\mathbf{div}^{(2)}\mathbf{M}+\mathbf{T}=-\mathbf{c}\quad \mathbf{M}\cdot\mathbf{n}=\mathbf{C}$ (44)

Global integrability condition for generalised strains.

As in the case of the 3-D continuum a particular interest should be given to selfequilibrated generalised strain fields, i.e., to those which are solutions of the above equations with null right-hand members. For plates also the following integrability condition can be proved (cf Nayroles):

Theorem 1

Let ξ be a defined field of generalised strains. There exists a generalised displacement field $\mathbf{q}$ such that:

$$\boldsymbol{\xi} = \mathbf{D}_{\mathrm{p}}\mathbf{q}$$

if and only if $\boldsymbol{\xi}$ is orthogonal to any self equilibrated field of generalised stresses, i.e.:

$$\forall \mathbf{Z} \in \mathbb{Z}_0: \quad \langle \boldsymbol{\xi}, \mathbf{Z} \rangle = 0$$

where $\mathbb{Z}_0$ denotes the vector-space of self equilibrated generalised stress fields i.e.:

$$\mathbb{Z}_0 = \{\mathbf{Z}/\forall \delta\mathbf{q}: \langle \mathbf{D}_{\mathrm{p}}\delta\mathbf{q}, \mathbf{Z} \rangle = 0\}$$

3. Equilibrium problems and constitutive equations

3.1. *Force laws*

We are going to consider external force laws, i.e. correspondences between the history $t \mapsto \tilde{\mathbf{U}}(t)$, of displacement $\mathbf{U}$ and that, $t \mapsto \tilde{\boldsymbol{\Phi}}(t)$, of external force $\boldsymbol{\Phi}$. As the linear operators used, $\mathbf{A}$, $\mathbf{B}^{+}$, $\mathbf{grad}_{\mathrm{s}}$ etc., are independent of time and of the real value of the variables (the transformations here are infinitesimal), it is probable that the general nature of what follows will not be markedly lessened by limiting ourselves to the correspondences between the current values of $\mathbf{U}$ and of $\boldsymbol{\Phi}$.

It may be a question of multivalued correspondences as in the case of constraints; therefore, $\mathscr{F}$ designating an external force law we choose to denote the correspondence:

$$\boldsymbol{\Phi} \in \mathscr{F}(\mathbf{U}) \tag{45}$$

where set $\mathscr{F}(\mathbf{q})$ is perhaps empty. The mechanical notion of force law is thus associated with the mathematical notion of multivalued function.

If many force laws reduce in fact to applications, $\mathscr{F}(\mathbf{q})$ being reduced to one point, constraint laws appear indeed as multivalued force laws. If we consider the case where the displacements compatible with the constraint, seen as a whole, are an affine variety $\mathbf{U}_0 + \mathbb{U}_1$: $\mathbf{U}_0$ is an imposed displacement, $\mathbb{U}_1$ a vector space. The hypothesis of infinitesimal transformations makes such a situation very common. If the constraint is "frictionless" the constraint force is arbitrary in the orthogonal of $\mathbb{U}_1$ and law $\mathscr{F}_{\mathrm{L}}$ which defines this constraint is:

$$\mathbf{U} \notin \mathbf{U}_0 + \mathbb{U}_1 \Rightarrow \mathscr{F}_{\mathrm{L}}(\mathbf{U}) = \varnothing$$
$$\mathbf{U} \in \mathbf{U}_0 + \mathbb{U}_1 \Rightarrow \mathscr{F}_{\mathrm{L}}(\mathbf{U}) = \{\boldsymbol{\Phi}_{\mathrm{L}}: \forall \delta\mathbf{U} \in \mathbb{U}_1 \langle\langle \delta\mathbf{U}, \boldsymbol{\Phi}_{\mathrm{L}} \rangle\rangle_3 = 0\} \tag{46}$$

It expresses simultaneously that the displacement belongs to the permitted set and that the constraint force is undetermined as long as its work vanishes in all virtual displacements compatible with the constraint.

Similarly, a generalised effort law is a multi-application which defines the correspondence:

$$\mathbf{Q} \in \mathscr{Q}(\mathbf{q}) \tag{47}$$

3.2. *Zero complementary virtual work forces*

The following proposition gives the general form of external forces of zero virtual work in all complementary displacements:

Proposition 4. Characterisation of zero complementary virtual work forces.

Let $\mathbf{\Phi} = (\mathbf{f}, \mathbf{F}^1, \mathbf{F}^2, \mathbf{F}^3)$ be an external force; we have the property:

$$\forall \delta\mathbf{u}: \quad \langle\langle \delta\mathbf{u}, \mathbf{\Phi} \rangle\rangle_3 = 0$$

if and only if

- $\mathbf{F}^1$ and $\mathbf{F}^2$ are zero;
- the components along x and y of $\mathbf{f}$ and of $\mathbf{F}^3$ are *affine* functions of z, the coefficients of which depend on (x, y):

$$f_1 = \lambda_1(x,y) + z\lambda_4(x,y); \quad F_1^3 = \mu_1(x,y) + z\mu_4(x,y) \tag{48}$$

$$f_2 = \lambda_2(x,y) + z\lambda_5(x,y); \quad F_2^3 = \mu_2(x,y) + z\mu_5(x,y) \tag{49}$$

- the components along z of $\mathbf{f}$ and of $\mathbf{F}^3$ depend only on (x, y):

$$f_3 = \lambda_3(x,y); \quad F_3^3 = \mu_3(x,y) \tag{50}$$

Proof of this is established in Nayroles. But it can already be seen that if $\mathbf{\Phi}$ is of the above form it is indeed of zero work in any displacement which complies with constraint equations (7); for the reverse to be true it is necessary to take some precautions in analysis. Coefficients λ_i and μ_i appear as Lagrange multipliers associated with constraints (7) imposed on complementary displacements.

Now if $\mathbf{\Phi}$ is of the form given by proposition 4 it produces the generalised effort:

$$\mathbf{Q} = (\mathbf{g}, p, \mathbf{c}, \mathbf{G}, P, \mathbf{C}) = \mathbf{A}^{+\mathrm{T}}\mathbf{\Phi}$$

with:

$$\begin{aligned} &\mathbf{g} = h\lambda_1\mathbf{i} + h\lambda_2\mathbf{j}; \quad p = h\lambda_3; \quad \mathbf{c} = \mathbf{I}\lambda_4\mathbf{i} + \mathbf{I}\lambda_5\mathbf{j} \\ &\mathbf{G} = h\mu_1\mathbf{i} + h\mu_2\mathbf{j}; \quad P = h\mu_3; \quad \mathbf{C} = \mathbf{I}\mu_4\mathbf{i} + \mathbf{I}\mu_5\mathbf{j} \end{aligned} \tag{51}$$

Conversely, any generalised effort $\mathbf{Q}$ is associated with a sole external force $\mathbf{\Phi}$ of the form given by proposition 4 for coefficients λ_i and μ_i easily calculated with the help of formulae (51). $\mathbf{A}^{\mathrm{T}}$ denotes the corresponding linear operator:

$$\mathbf{\Phi} = \mathbf{A}^{\mathrm{T}}\mathbf{Q}$$

which obviously verifies:

$$\forall \mathbf{Q}: \quad \mathbf{A}^{+\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{Q} = \mathbf{Q}$$

The following transposition equality is then derived from (16):

$$\forall \delta\mathbf{V}, \forall \mathbf{Q}: \quad \langle\langle \delta\mathbf{V}, \mathbf{A}^{\mathrm{T}}\mathbf{Q} \rangle\rangle = \langle\langle \mathbf{A}\delta\mathbf{V}, \mathbf{Q} \rangle\rangle_3$$

and justifies notation $\mathbf{A}^{\mathrm{T}}$. At last $\mathbf{A}^{+\mathrm{T}}$ is the right-hand inverse:

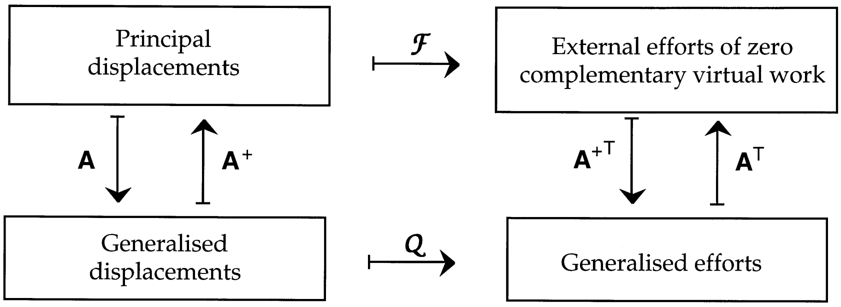

Fig. 2. Displacement and effort functional spaces and the operators connecting them.

$$\mathbf{A}^{T}\mathbf{A}^{+T} = \mathbf{I}$$

The situation can then be described by the following diagram:
where multiapplications $\mathscr{F}$ and $\mathscr{Q}$ designate laws of external force and of generalised effort, the equivalence of which we are now going to study.

3.3. *Equivalence between a law of external force and a law of generalised effort*

Let us now return to equilibrium equations (41), (42) and to proposition 2. We are trying to discover whether force laws (45) and (47) can be equivalent, in the sense of equivalence between systems (41.1)+(45) and (42)+(47) when variables $\mathbf{V}$, $\mathbf{q}$, $\mathbf{\Phi}$, $\mathbf{Q}$ are constrained by the habitual relations:

$$\mathbf{q} = \mathbf{A}\mathbf{V}; \quad \mathbf{Q} = \mathbf{A}^{+T}\mathbf{\Phi} \tag{52}$$

In the following definition two properties, which are obviously necessary for this equivalence to exist, are combined to make one, named property $\mathscr{P}$ (like the first letter of the word "plate"). The first is that the force is of zero complementary virtual work. The second is that the generalised effort it produces is independent of the complementary displacement, i.e.:

$$\forall\mathbf{V}\forall\mathbf{u}:(\mathbf{A}^{+T}\circ\mathscr{F})(\mathbf{V}+\mathbf{u}) = \mathbf{A}^{+T}\circ\mathscr{F})(\mathbf{V})$$

But then law $\mathscr{F}$ is independent of $\mathbf{u}$ since for all virtual displacements we have:

$$\langle\langle\delta\mathbf{V}+\delta\mathbf{u},\mathscr{F}(\mathbf{V}+\mathbf{u})\rangle\rangle_3 = \langle\langle\delta\mathbf{V},\mathscr{F}(\mathbf{V}+\mathbf{u})\rangle\rangle_3 = \langle\langle\delta\mathbf{q},\mathbf{A}^{+T}\mathscr{F}(\mathbf{V}+\mathbf{u})\rangle\rangle$$
$$= \langle\langle\delta\mathbf{q},\mathbf{A}^{+T}\mathscr{F}(\mathbf{V})\rangle\rangle = \langle\langle\delta\mathbf{V},\mathscr{F}(\mathbf{V})\rangle\rangle_3 = \langle\langle\delta\mathbf{V}+\delta\mathbf{u},\mathscr{F}(\mathbf{V})\rangle\rangle_3$$

We are, thus, led to pose the following definition:

Definition 1. Property $\mathscr{P}$ of an external force law.
An external force law $\mathscr{F}$ possesses property $\mathscr{P}$ if, by definition:

1. the external force it defines does not work in virtual displacements of the complementary type, i.e.:

$$\forall \delta \mathbf{u}: \quad \langle\langle \delta \mathbf{u}, \mathscr{F}(\mathbf{V}+\mathbf{u}) \rangle\rangle_3 = 0 \tag{53}$$

2. it is independent of the complementary displacement, i.e.:

$$\forall \mathbf{V} \forall \mathbf{u}: \quad \mathscr{F}(\mathbf{V}+\mathbf{u}) = \mathscr{F}(\mathbf{V}) \tag{54}$$

Proposition 5

The external force law $\mathscr{F}$ possesses property $\mathscr{P}$ if and only if it is equivalent to the generalised effort law $\mathscr{Q}$ defined by:

$$\mathscr{Q} = \mathbf{A}^{+\mathrm{T}} \circ \mathscr{F} \circ \mathbf{A}^{+} \tag{55}$$

which means that conversely:

$$\mathscr{F} = \mathbf{A}^{\mathrm{T}} \circ \mathscr{Q} \circ \mathbf{A} \tag{56}$$

Indeed, notice that (56) implies (55), since $\mathbf{A}^{+}$ and $\mathbf{A}^{\mathrm{T}}$ are respective right-hand inverses of $\mathbf{A}$ and $\mathbf{A}^{+\mathrm{T}}$; now if force law $\mathscr{F}$ possesses property $\mathscr{P}$ then Fig. 2 shows that (53) and (54) can be written, respectively:

$$\Leftrightarrow \mathscr{F} = \mathbf{A}^{\mathrm{T}} \circ \mathbf{A}^{+\mathrm{T}} \circ \mathscr{F} \tag{53}$$

$$\Leftrightarrow \mathscr{F} = \mathscr{F} \circ \mathbf{A}^{+} \circ \mathbf{A} \tag{54}$$

which yield (56). Conversely (56) implies (53) and (54), which ends the proof.

Now let us consider constraint law $\mathscr{F}_{\mathrm{L}}$ defined in (46). It is proved that $\mathscr{F}_{\mathrm{L}}$ possesses property $\mathscr{P}$ if and only if $\mathbb{U}_1$ contains all the complementary displacements. If we then pose:

$$\mathbf{q}_0 = \mathbf{A}\mathbf{U}_0 \quad \text{and} \quad \mathbb{W}_1 = \mathbf{A}(\mathbb{U}_1) \tag{57}$$

$\mathscr{F}_{\mathrm{L}}$ is equivalent to the frictionless constraint law $\mathscr{Q}_{\mathrm{L}}$ associates with the affine manifold $\mathbf{q}_0 + \mathbb{W}_1$:

$$\mathbf{q} \notin \mathbf{q}_0 + \mathbb{W}_1 \Rightarrow \mathscr{Q}_{\mathrm{L}}(\mathbf{q}) = \varnothing$$

$$\mathbf{q} \in \mathbf{q}_0 + \mathbb{W}_1 \Rightarrow \mathscr{Q}_{\mathrm{L}}(\mathbf{q}) = \{\mathbf{Q}_{\mathrm{L}}: \forall \delta \mathbf{q} \in \mathbb{W}_1 \langle\langle \delta \mathbf{q}, \mathbf{Q}_{\mathrm{L}} \rangle\rangle_3 = 0\} \tag{58}$$

The 3-D-constitutive laws envisaged are also multivocal correspondences which will be written:

$$\boldsymbol{\sigma} \in \mathscr{K}_3(\boldsymbol{\varepsilon}) \tag{59}$$

limiting ourselves, as in the case of external effort laws, to correspondences between the values of the strain and of the stress at the moment under consideration.

Such as it has just been described, this law can be a correspondence between fields, i.e., a "non-local" law. In fact, we shall assume later that it is local, i.e., of the form:

$$\boldsymbol{\sigma}(P) \in \mathscr{K}_3(\boldsymbol{\varepsilon}(P), P) \tag{60}$$

where P designates the point (x, y, z).

3.4. *Equilibrium problem of the* 3-*D model and plate problem*

System (41) is that of the equilibrium equations of an ordinary 3-D solid, but whose kinematic description contains that of plates. In order to pose a problem of equilibrium it is necessary to combine effort laws with this equation: the laws of external forces, and the material constitutive law. The constraint laws, or, if preferred, support conditions of the solid, are obviously particular external effort laws. The corresponding system can be written:

$$\forall \delta \mathbf{V}: \quad \langle\langle \delta \mathbf{V}, \mathbf{\Phi} \rangle\rangle_3 + \langle \mathbf{grad}_s \delta \mathbf{V}, \boldsymbol{\sigma} \rangle_3 = 0 \quad \text{principal equilibrium equ.} \tag{61.1}$$

$$\mathbf{\Phi} \in \mathbf{\Phi}(\mathbf{V} + \mathbf{u}) \quad \text{external force law} \tag{61.2}$$

$$\forall \delta \mathbf{u}: \quad \langle\langle \delta \mathbf{u}, \mathbf{\Phi} \rangle\rangle_3 + \langle \mathbf{grad}_s \delta \mathbf{u}, \boldsymbol{\sigma} \rangle_3 = 0 \quad \text{complementary equilibrium equ.} \tag{61.3}$$

$$\boldsymbol{\varepsilon} = \mathbf{grad}_s(\mathbf{V} + \mathbf{u}) + \boldsymbol{\varepsilon}_0 \quad \text{displacement–strain equ.} \tag{61.4}$$

$$\boldsymbol{\sigma} \in \mathscr{K}_3(\boldsymbol{\varepsilon}) \quad \text{3-D-constitutive law} \tag{61.5}$$

This formulation requires commentary.

Quantity $\boldsymbol{\varepsilon}_0$ designates a field of imposed strains, which may be of thermal origin[1] for example, but which may also result from a displacement imposed by constraints.

Letter $\mathbf{\Phi}$ designates the sum of all the external efforts, let us say $\mathbf{\Phi}_i$, each being governed by a law $\mathscr{F}_i$. Law $\mathscr{F}$ is the sum of laws $\mathscr{F}_i$, in other words, set $\mathscr{F}(\mathbf{V}+\mathbf{u})$ is the direct sum of sets $\mathscr{F}_i(\mathbf{V}+\mathbf{u})$.

Among the $\mathscr{F}_i$ are the constraint relations which define the load conditions.

The objective of the theory is to replace the 3-D equations of system (61) by a bidimensional system which is concerned only by generalised variables, i.e., of the following form:

$$\forall \delta \mathbf{q}: \quad \langle\langle \delta \mathbf{q}, \mathbf{Q} \rangle\rangle + \langle \mathbf{D}_p \delta \mathbf{q}, \mathbf{Z} \rangle = 0 \quad \text{plate equilibrium equ.} \tag{62.1}$$

$$\mathbf{Q} \in \mathscr{Q}(\mathbf{q}) \quad \text{generalised effort law} \tag{62.2}$$

$$\boldsymbol{\xi} = \mathbf{D}_p \mathbf{q} + \boldsymbol{\xi}_0 \quad \text{gen. disp.–gen. def. relation} \tag{62.3}$$

$$\mathbf{Z} \in \mathscr{K}_p(\boldsymbol{\xi}) \quad \text{plate constitutive law} \tag{62.4}$$

At this point it is irrelevant to know whether $\mathscr{K}_p$ is local or not: it is merely a correspondence between generalised strain fields and generalised stress fields.

Let us attend first of all to determining the conditions by which system (62) arises from system (61).

Equation (62.1), as was seen in proposition 2, is equivalent to eqn (61.1) when $\mathbf{Z} = \mathbf{B}^{+\mathrm{T}}\boldsymbol{\sigma}$ is the generalised stress produced by $\boldsymbol{\sigma}$ and $\mathbf{Q} = \mathbf{A}^{+\mathrm{T}}\mathbf{\Phi}$ the generalised effort produced by $\mathbf{\Phi}$.

In order that eqn (61.2) give rise to eqn (62.2) it is necessary and sufficient that $\mathscr{F}$ possesses the following property:

$$\forall \mathbf{u}: \quad (\mathbf{A}^{+\mathrm{T}} \circ \mathscr{F})(\mathbf{V} + \mathbf{u}) = (\mathbf{A}^{+\mathrm{T}} \circ) \mathscr{F}(\mathbf{V}) \tag{63}$$

and the value of $\mathscr{Q}$ is then $\mathbf{A}^{+\mathrm{T}} \circ \mathscr{F} \circ \mathbf{A}^{+}$.

To return to relation (31), which links principal strain $\mathbf{e}$ according to generalised strain $\boldsymbol{\xi}$, it is seen that (61.4) brings about:

[1] We shall choose $\varepsilon^0 = -\alpha\theta$, where $\alpha\theta$ is the thermal dilatation tensor at temperature θ, which allows us to keep constitutive law $\mathscr{K}_3$ when it does not depend on θ.

$$\boldsymbol{\varepsilon} = \mathbf{B}^{+}\xi + \mathbf{grad}_{s}\mathbf{u} + \boldsymbol{\varepsilon}_0 \tag{64}$$

where ξ, as in (28), is equal to $\mathbf{D}_p\mathbf{q}$, i.e. $\mathbf{D}_p\mathbf{AV}$.

Taking (64) into account, eqns (61.3) and (61.5) are now written:

$$\forall\delta\mathbf{u}: \quad \langle\langle\delta\mathbf{u}, \mathbf{\Phi}\rangle\rangle_3 + \langle\mathbf{grad}_s\delta\mathbf{u}, \boldsymbol{\sigma}\rangle_3 = 0 \tag{65.1}$$

$$\boldsymbol{\sigma} \in \mathscr{K}_3(\mathbf{B}^{+}\xi + \mathbf{grad}_s\mathbf{u} + \boldsymbol{\varepsilon}_0) \tag{65.2}$$

Fields $\mathbf{u}$ and $\boldsymbol{\sigma}$ are, for $\mathbf{\Phi}$ given, the unknowns of this system. For its possible solutions to depend only on ξ the virtual work $\langle\langle\delta\mathbf{u}, \mathbf{\Phi}\rangle\rangle_3$ must be zero; therefore, law $\mathscr{F}$ must possess the following property:

$$\mathbf{\Phi} \in \mathscr{F}(\mathbf{V} + \mathbf{u}) \Rightarrow \langle\langle\delta\mathbf{u}, \mathbf{\Phi}\rangle\rangle_3 = 0 \tag{66}$$

So it is necessary that law $\mathscr{F}$ possesses both property (63) and (66), i.e. property $\mathscr{P}$, and that it therefore be equivalent to the law of generalised effort denoted $\mathscr{Q}$ defined in (55).

It is also convenient, although not indispensable to the present theory, that the solution of (65) should not depend on the 3-D field $\boldsymbol{\varepsilon}_0$, which varies from one case to another, but rather on a 2-D field ξ_0 as in (62.3). Accordingly, $\boldsymbol{\varepsilon}_0$ is attributed the form:

$$\boldsymbol{\varepsilon}_0 = \mathbf{B}^{+}\xi_0 \tag{67}$$

which, now using definition (62.3) of ξ, enables (64) to be written:

$$\boldsymbol{\varepsilon} = \mathbf{B}^{+}\xi + \mathbf{grad}_s\mathbf{u}$$

where no imposed strain remains.

Under these conditions, system (65) appears in its final form:

$$\boxed{\boldsymbol{\sigma} \in \mathscr{K}_3(\mathbf{B}^{+}\xi + \mathbf{grad}_s\mathbf{u}) \quad \forall\delta\mathbf{u}: \quad \langle\mathbf{grad}_s\delta\mathbf{u}, \boldsymbol{\sigma}\rangle_3 = 0} \tag{68}$$

pair $(\mathbf{u}, \boldsymbol{\sigma})$ being the unknown.

$\mathscr{C}(\xi)$ denotes the ensemble, which may be empty, of the solutions of (68), $\mathscr{U}(\xi)$ and $\mathscr{S}(\xi)$ its projections on the spaces of the $\mathbf{u}$'s and the $\boldsymbol{\sigma}$'s:

$$\mathbf{u} \in \mathscr{U}(\xi) \Leftrightarrow \exists\,\boldsymbol{\sigma}: \quad (\mathbf{u}, \boldsymbol{\sigma}) \in \mathscr{C}(\xi)$$

$$\boldsymbol{\sigma} \in \mathscr{S}(\xi) \Leftrightarrow \exists\,\mathbf{u}: \quad (\mathbf{u}, \boldsymbol{\sigma}) \in \mathscr{C}(\xi)$$

The plate constitutive law can then be written in the form:

$$\mathscr{K}_p = \mathbf{B}^{+T}\mathscr{S} \tag{69}$$

It was, thus, supposed that external force law $\mathscr{F}$ possesses property $\mathscr{P}$ and, also, that the field of imposed strains $\boldsymbol{\varepsilon}_0$ has the form (67). In fact we have demonstrated that if quintuplet $(\mathbf{V}, \mathbf{u}, \mathbf{\Phi}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})$ solves the 3-D problem (61), then the quadruplet $(\mathbf{q}, \mathbf{Q}, \xi, \mathbf{Z})$ solves plate problem (62) for:

$$\mathbf{q} = \mathbf{AV}; \quad \mathbf{Q} = \mathbf{A}^{+T}\mathbf{\Phi}; \quad \xi = \mathbf{D}_p\mathbf{AV}; \quad \mathbf{Z} = \mathbf{B}^{+T}\boldsymbol{\sigma}$$

and for the plate constitutive law given in (69), from the solutions of fundamental equation (68).

Inversely, let $(\mathbf{q}, \mathbf{Q}, \xi, \mathbf{Z})$ be a solution of problem (62) for this plate constitutive law. The equivalence, temporarily admitted, between laws $\mathscr{F}$ and $\mathscr{Q}$ means that there exists a force $\mathbf{\Phi}$, of zero complementary virtual work, such that (61.2) be satisfied for $\mathbf{V} = \mathbf{A}^{+}\mathbf{q}$ and any $\mathbf{u}$, and finally such that $\mathbf{Q} = \mathbf{A}^{+\mathrm{T}}\mathbf{\Phi}$.

It follows that, as $\boldsymbol{\xi}$ and $\mathbf{Z}$ satisfy (62.4), there exists a pair $(\mathbf{u}, \boldsymbol{\sigma})$, solution of (68) and such that $\mathbf{Z} = \mathbf{B}^{+\mathrm{T}}\boldsymbol{\sigma}$. As $\mathbf{Q}$ and $\mathbf{Z}$ are in equilibrium in accordance with (62.1) external effort $\mathbf{\Phi}$ and stress $\boldsymbol{\sigma}$ prove the principal equilibrium equation (61.1).

According to (62.3) $\boldsymbol{\xi}$ equals:

$$\boldsymbol{\xi} = \mathbf{D}_{\mathrm{p}}\mathbf{q} + \boldsymbol{\xi}_0 = \mathbf{D}_{\mathrm{p}}\mathbf{A}\mathbf{V} + \boldsymbol{\xi}_0$$

so that when we write:

$$\boldsymbol{\varepsilon} = \mathbf{B}^{+}\boldsymbol{\xi} + \mathbf{grad}_{\mathrm{s}}\mathbf{u} \quad \text{and} \quad \boldsymbol{\varepsilon}_0 = \mathbf{B}^{+}\boldsymbol{\xi}_0$$

field $\boldsymbol{\varepsilon}$ proves (64) and pair $(\boldsymbol{\varepsilon}, \boldsymbol{\sigma})$ verifies constitutive law (61.5).

Finally, as $(\mathbf{u}, \boldsymbol{\sigma})$ solves (68) and $\mathbf{\Phi}$ is of zero complementary work, eqn (61.3) is also satisfied.

In the end, quintuplet $(\mathbf{V}, \mathbf{u}, \mathbf{\Phi}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})$ solves the 3-D problem, and the following theorem can be stated:

Theorem 2. Equivalence between the plate problem and the 3-D problem.

If the external effort laws are equivalent to generalised effort laws (property $\mathscr{P}$), if the field of strains $\boldsymbol{\varepsilon}_0$ imposed on the 3-D milieu is of the form (67), then there exists a global plate constitutive law $\mathscr{K}_{\mathrm{p}}$ obtained by solving eqn (68) and for which plate problem (62) is equivalent to the 3-D problem (61).

Unfortunately, the constitutive law thus determined is obviously not local. Dependent in particular on the plate contour, it is useless. The following paragraph will define approximations of this law by means of local laws.

3.5. *Effort laws and Saint-Venant's principle*

It was supposed that external effort law $\mathscr{F}$ is equivalent to generalised effort law $\mathscr{Q}$; to what degree is this hypothesis valid? Let us examine the various types of effort which can enter $\mathscr{F}$.

The volume efforts exerted on a solid very often possesses, in practice, property $\mathscr{P}$. This is the case for gravity in a homogeneous medium. On the contrary, laws concerning effort forces exerted on the upper or lower faces obviously do not possess $\mathscr{P}$.

When any external effort law $\mathscr{F}$ is to be taken into consideration, an approximation must be made by replacing it with a law $\mathscr{F}$ which possesses $\mathscr{P}$, by using the formula:

$$\mathscr{F}(\mathbf{V} + \mathbf{u}) = \mathbf{A}^{\mathrm{T}}\mathbf{A}^{+\mathrm{T}}\mathscr{F}(\mathbf{V} + \mathbf{u}_0)$$

where $\mathbf{u}_0$ is a fixed complementary field, chosen arbitrarily and, hopefully, wisely, which is usually zero. The corresponding generalised effort law is then given by:

$$\mathbf{q} \mapsto \mathscr{Q}(\mathbf{q}) = \mathbf{A}^{+\mathrm{T}}\mathscr{F}(\mathbf{A}^{+}\mathbf{q} + \mathbf{u}_0)$$

The quality of the approximation obtained depends on the properties of law $\mathscr{F}$, but also on the solid's behaviour as a whole. In any case, replacing $\mathscr{F}$ by a law $\mathscr{F}$, which gives rise to the same generalised effort $\mathbf{Q}$, is in agreement with what P. Ladevèze [P.Lad.1.] calls Saint-Venant's principle "in effort"; in parallel, choosing a particular $\mathbf{u}_0$, to evaluate $\mathscr{F}(\mathbf{V}+\mathbf{u})$, may be regarded as an expression of Saint-Venant's principle in terms of displacements (different from what the same author calls "Saint-Venant's principle for displacements").

Similarly, this principle also applies to the approximation of a field of imposed strains $\boldsymbol{\varepsilon}_0$ by a field of the form $\mathbf{B}^+\boldsymbol{\xi}_0$.

The supporting efforts practically never possess $\mathscr{P}$ since the constraint should only concern the principal part of the displacement. Yet, while the latter possesses a mathematical reality, it has no material reality except perhaps in exceptional circumstances. Physically, constraint efforts are almost always contact efforts which depend on the total displacement of the points they concern. What has just been said regarding external efforts and the generalised strains associated with them, remains particularly true for constraint efforts. However, most commonly, we give ourselves the constraint conditions of a plate directly in terms of generalised displacements and strains, as shown in (58), without considering how they could appear in the 3-D context.

4. Constitutive laws and polynomial solutions

4.1. *General method*

We return to the fundamental eqn (68) which defines the couple of unknown fields $(\mathbf{u}, \boldsymbol{\sigma})$ according to the field of generalised strains $\boldsymbol{\xi}$. This "variational" system defines a boundary value problem. The global constitutive law $\mathscr{K}_\mathrm{p}$ deduced from it is "accurate", in the sense that it enables us to write the system of plate equations equivalent to the system of the 3-D solid equations. As said before, it presents two major shortcomings: it is not local and it depends on the contour.

The first step will be to overcome the second of these drawbacks by using, in (68), not all the test fields $\delta\mathbf{u}$, but only those whose support is of projection on $(x, 0, y)$ contained in the open set Ω. It will be convenient to use the following notations:

Notations

Let $\mathbb{U}^\mathrm{c}$ be the vectorial space of the complementary fields.

Let $(\mathbb{U}^\mathrm{c})^0$ be its orthogonal, i.e., the space of the external efforts which are of zero virtual work in all virtual displacement belonging to $\mathbb{U}^\mathrm{c}$.

Let $\mathbb{U}^\mathrm{c}_0$ be the vectorial subspace of the complementary fields which vanish outside a cylinder $\mathbf{K}\times\mathbb{R}$, where $\mathbf{K}$ is a compact set contained in Ω.

Let $(\mathbb{U}^\mathrm{c}_0)^0$ be its orthogonal, i.e., the space of the external efforts which are of zero virtual work in any virtual displacement belonging to $\mathbb{U}^\mathrm{c}_0$.

Equation (68) is then replaced by:

$$\boxed{\boldsymbol{\sigma}\in\mathscr{K}_3(\mathbf{B}^+\boldsymbol{\xi}+\mathbf{grad}_\mathrm{s}\mathbf{u}), \quad \forall\delta\mathbf{u}\in\mathbb{U}^\mathrm{c}_0: \quad \langle\mathbf{grad}_\mathrm{s}\delta\mathbf{u}, \boldsymbol{\sigma}\rangle_3 = 0} \tag{70}$$

leading, in a sense, to a constitutive equation in the interior, or, in other words, in the undefined medium.

In order to deduce from this equation a useful differential system we first establish the following proposition, which completes proposition 4:

Proposition 6

Orthogonal $(\mathbb{U}_0^c)^0$ is the space of efforts $\mathbf{\Phi} = (\mathbf{f}, \mathbf{F}^1, \mathbf{F}^2, \mathbf{F}^3)$ such that:

- $\mathbf{F}^1$ and $\mathbf{F}^2$ are zero;
- components f_1 and f_2 of $\mathbf{f}$ are affine functions of z, of which the coefficients depend on (x, y) while f_3 is independent of z:

$$f_1 = \lambda_1(x,y) + z\lambda_4(x,y); \quad f_2 = \lambda_2(x,y) + z\lambda_5(x,y); \quad f_3 = \lambda_3(x,y) \tag{71}$$

Component $\mathbf{F}^3$ is obviously not concerned by $\mathbb{U}_0^c$: it can be anything. To demonstrate this proposition we start with the orthogonal space definition:

$$\mathbf{\Phi} \in (\mathbb{U}_0^c)^0 \Leftrightarrow \forall \delta\mathbf{u} \in \mathbb{U}_0^c: \quad \int_{\mathbf{\Psi}} \mathbf{f} \cdot \delta\mathbf{u} \, \mathrm{d}V + \int_{\partial_1 \mathbf{\Psi}} \mathbf{F}_1 \cdot \delta\mathbf{u} \, \mathrm{d}S + \int_{\partial_2 \mathbf{\Psi}} \mathbf{F}_2 \cdot \delta\mathbf{u} \, \mathrm{d}S = 0 \tag{72}$$

The $\delta\mathbf{u}$ verify the constraint eqns (7) in such a way that if $\mathbf{F}_1$ and $\mathbf{F}_2$ are zero and $\mathbf{f}$ of the form (71) then $\mathbf{\Phi}$ effectively belongs to $(\mathbb{U}_0^c)^0$.

To show that the reverse is also true let us consider an open sub-domain $\mathbf{\Omega}_\varepsilon$ whose adherence is contained in $\mathbf{\Omega}$ and whose measure (area) tends towards that of $\mathbf{\Omega}$ when ε tends towards 0.

Let us write:

$$\mathbf{\Psi}_\varepsilon = \mathbf{\Psi} \bigcap (\mathbf{\Omega}_\varepsilon \times \mathbb{R})$$
$$\partial_1 \mathbf{\Psi}_\varepsilon = \partial_1 \mathbf{\Psi} \bigcap (\mathbf{\Omega}_\varepsilon \times \mathbb{R})$$
$$\partial_2 \mathbf{\Psi}_\varepsilon = \partial_2 \mathbf{\Psi} \bigcap (\mathbf{\Omega}_\varepsilon \times \mathbb{R})$$

Let $\delta\mathbf{u}$ be a field belonging to $\mathbb{U}^c$, let $\delta\mathbf{u}_\varepsilon$ be its restriction at $\mathbf{\Psi}_\varepsilon$. We have:

$$\forall \delta\mathbf{u} \in \mathbb{U}^c \, \exists \, \delta\mathbf{u}' \in \mathbb{U}_0^c: \quad \delta\mathbf{u}_\varepsilon = \delta\mathbf{u}'_\varepsilon$$

Then:

$$\mathbf{\Phi} \in (\mathbb{U}_0^c)^0 \Rightarrow \forall \delta\mathbf{u} \in \mathbb{U}^c: \quad 0 = \langle \delta\mathbf{u}', \mathbf{\Phi} \rangle = \int_{\mathbf{\Psi}_\varepsilon} \mathbf{f} \cdot \delta\mathbf{u} \, \mathrm{d}V + \int_{\partial_1 \mathbf{\Psi}_\varepsilon} \mathbf{F}_1 \cdot \delta\mathbf{u} \, \mathrm{d}\Sigma + \int_{\partial_2 \mathbf{\Psi}_\varepsilon} \mathbf{F}_2 \cdot \delta\mathbf{u} \, \mathrm{d}\Sigma$$
$$+ \int_{\mathbf{\Psi} - \mathbf{\Psi}_\varepsilon} \mathbf{f} \cdot \delta\mathbf{u}' \, \mathrm{d}V + \int_{\partial_1 \mathbf{\Psi} - \partial_1} \mathbf{F}_1 \cdot \delta\mathbf{u}' \, \mathrm{d}\Sigma + \int_{\partial_2 \mathbf{\Psi} - \partial_2 \mathbf{\Psi}_\varepsilon} \mathbf{F}_2 \cdot \delta\mathbf{u}' \, \mathrm{d}\Sigma$$

The second line of this sum tends towards zero and the first towards $\langle \delta\mathbf{u}, (\mathbf{f}, \mathbf{F}_1, \mathbf{F}_2, \mathbf{0}) \rangle$. We thus obtain:

$$\mathbf{\Phi} \in (\mathbb{U}_0^c)^0 \Rightarrow (\mathbf{f}, \mathbf{F}_1, \mathbf{F}_2, 0) \in (\mathbb{U}^c)^0$$

By using proposition 4 it can be deduced that $\mathbf{f}$, $\mathbf{F}_1$ and $\mathbf{F}_2$ are indeed of the form announced by proposition 6.

By using, classically, the identity:

$$\langle \mathbf{grad}_s \delta\mathbf{u}, \boldsymbol{\sigma} \rangle_3 = \int_{\Psi} \mathbf{grad}_s \delta\mathbf{u} : \quad \boldsymbol{\sigma}\, \mathrm{d}V = -\int_{\Psi} \delta\mathbf{u} \cdot \mathbf{div}\, \boldsymbol{\sigma}\, \mathrm{d}V + \int_{\partial\Psi} \delta\mathbf{u} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}\, \mathrm{d}\Sigma$$

we arrive easily at

Proposition 7

Equation (70) is equivalent to the system:

$$\begin{cases} \mathbf{div}\, [\mathscr{K}_3 (\mathbf{B}^+ \boldsymbol{\xi} + \mathbf{grad}_s \mathbf{u}) \ni (\lambda_1 + \lambda_4 z)\mathbf{i} + (\lambda_2 + \lambda_5 z)\mathbf{j} + \lambda_3 \mathbf{k} & (73.1) \\ \sigma\left(x, y, \dfrac{h}{2}\right) \cdot \mathbf{n}^2 \left(x, y, \dfrac{h}{2}\right) = \sigma\left(x, y, -\dfrac{h}{2}\right) \cdot \mathbf{n}^1 \left(x, y, -\dfrac{h}{2}\right) = 0 & (73.2) \\ \displaystyle\int_{-(h/2)}^{h/2} \mathbf{u}\, \mathrm{d}z = 0; \quad \int_{-(h/2)}^{h/2} u_1 z\, \mathrm{d}z = 0; \quad \int_{-(h/2)}^{h/2} u_2 z\, \mathrm{d}z = 0 & (73.3) \end{cases}$$

where $\mathbf{n}^1$ and $\mathbf{n}^2$ designate respectively the unit vectors normal to surfaces $\partial_1 \boldsymbol{\Psi}$ and $\partial_2 \boldsymbol{\Psi}$.

Considered from the angle of the single variable z this system appears as a differential equation of the second order (73.1) where, in the second member, the five auxiliary unknowns λ_i are found. The two boundary conditions (73.2) and the five scalar constraint equations (73.3) complete the system. The end of this paragraph will be devoted to some common situations where the latter can be solved: those where the constitutive law is linear and the field of generalised strains polynomial.

The manner in which this system has been constructed, considering only test fields which are zero at their edge, guarantees no validity of results in the vicinity of the latter. Far from being surprising, this restriction is rather reassuring in the sense that it has been known for a long time that plate theories are only valid at a sufficient distance from the edge, meaning at a distance approximately equivalent to the thickness of the plate.

4.2. *Polynomial solutions and accurate laws for finite elements*

From this point on, we assume that constitutive law $\mathscr{K}_3$ is a local application which gives the stress tensor according to that of the strains:

$$\boldsymbol{\sigma}(P) = \mathscr{K}_3(\boldsymbol{\varepsilon}(P), P) \quad \text{where } P = (x, y, z) \tag{74}$$

In eqn (73.1) symbol "$\ni$" should obviously be replaced by "$=$".

Preliminary remarks

In this paragraph we impose particular generalized strains, i.e. polynomials of degree 2 with respect to (x, y). The solutions of system (72) are then sought. Doing this we have implicitly introduced constraints; therefore, equilibrium equations are not verified and their residues can be identified to the efforts associated with these constraints.

Moreover, imposed generalised strains are any polynomials of degree 2, which need not verify compatibility eqns (32) and (33): the results thus obtained can be applied to problems where a generalised strain $\boldsymbol{\xi}_0$ is imposed, as indicated before.

Ending this paragraph we shall study solutions of Saint-Venant's type, i.e. solutions of the flexion problem for unloaded areas of the plate.

4.2.1. *Constitutive equation*

If an approximate plate local constitutive law $\mathscr{K}_{\text{app}}$ is sought, it is logical to consider a state of the plate in which the field of generalised strains $\boldsymbol{\xi}$ is constant, and to look for a complementary displacement $\mathbf{u}$ which is a function only of z, and a column $\boldsymbol{\lambda}$ of the Lagrange multipliers λ_i which is constant. Equation (73.1) then becomes an ordinary differential equation of the second order in z and, in general, system (73) allows a single solution $(\mathbf{u}^*, \boldsymbol{\sigma}^*, \boldsymbol{\lambda}^*)$ which depends on $\boldsymbol{\xi}$, and which can be written:

$$\boldsymbol{\sigma}^* = \mathscr{S}(\xi) \tag{75}$$

The generalised stress $\mathbf{Z}^*$ is calculated by formulae (37) from $\boldsymbol{\sigma}^*$, thus giving the local plate constitutive equation:

$$\mathscr{K}_{\text{app}} = \mathbf{B}^{+\mathrm{T}}\mathscr{L} \tag{76}$$

Since eqn (68) has been replaced by eqn (70), equivalent to system (73), and $\mathbf{u}^*$ does not belong to $\mathbb{U}_0^{\mathrm{c}}$, there is no "global" reason why work $\langle \mathbf{grad}_{\mathrm{s}}\mathbf{u}^*, \boldsymbol{\sigma}^*\rangle_3$ should be zero. However, since $\mathbf{u}^*$ is a function only for z we have:

$$\boldsymbol{\sigma}^*\text{:}\quad \mathbf{grad}_{\mathrm{s}}\mathbf{u}^* = \frac{\partial}{\partial z}\sum_i(\sigma_{i3}^* u_i^*) - \mathbf{u}^* \cdot \mathbf{div}\,\boldsymbol{\sigma}^* \tag{77}$$

which yields:

$$\langle \mathbf{grad}_{\mathrm{s}}\mathbf{u}^*, \boldsymbol{\sigma}^*\rangle_3 = \int_{\boldsymbol{\Omega}} \left\{ \int_{-(h/2)}^{h/2} \left[\frac{\partial}{\partial z}\sum_i(\sigma_{i3}^* u_i^*) - \mathbf{u}^* \cdot \mathbf{div}\,\boldsymbol{\sigma}^* \right] \mathrm{d}z \right\} \mathrm{d}\Sigma$$

which vanishes. Indeed, as $(\mathbf{u}^*, \boldsymbol{\sigma}^*, \boldsymbol{\lambda}^*)$ is a solution of (73), the first term of the bracket gives an integral on z which vanishes because of the boundary conditions, and so does the second because $\mathbf{u}^*$ is a complementary field, while $\mathbf{div}\,\boldsymbol{\sigma}^*$ verifies (73.1). The work of stress field $\boldsymbol{\sigma}^*$ in the total strain field is thus reduced to:

$$\langle \mathbf{grad}_{\mathrm{s}}(\mathbf{V} + \mathbf{u}^*), \boldsymbol{\sigma}^*\rangle_3 = \langle \xi, \mathscr{L}(\xi)\rangle \tag{78}$$

In particular, if the law of 3-D behaviour is the linear elasticity, the elastic energy of the 3-D stress field equals that of the generalised stress field.

4.2.2. *Polynomial solutions*

More generally, one may be seeking a solution $(\mathbf{u}^*, \boldsymbol{\sigma}^*, \boldsymbol{\lambda}^*)$ of system (73) where field $\boldsymbol{\xi}$ is of a given type, in particular polynomial. We have verified, to the second degree and for a plate of constant thickness, the following conjecture:

Conjecture

If the 3-D constitutive law is the linear hyper-elasticity,[2] if the field of generalised strains $\boldsymbol{\xi}$ is polynomial of d degree in (x, y), then solution $(\mathbf{u}^*, \boldsymbol{\sigma}^*, \boldsymbol{\lambda}^*)$ is polynomial in (x, y, z), of d degree in

[2] That is, there exists an elastic potential, or, in other words, stiffness matrix is symmetrical.

(x, y), and $\mathbf{u}^*$ is a polynome of $d+3$ degree with regard to z. This conjecture has been verified to $d = 2$.

The demonstration of this result is perhaps within the reach of a proficient user of formal calculation methods; we shall explain later the procedure adopted for $d = 2$ using "Mathematica".

Let us write, for $d = 1$, for example:

$$\boldsymbol{\xi} = \boldsymbol{\xi}^0 + \boldsymbol{\xi}^1 x + \boldsymbol{\xi}^2 y = \Xi^{\mathrm{T}} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} \quad \text{with:} \quad \Xi = \begin{Bmatrix} \boldsymbol{\xi}^0 \\ \boldsymbol{\xi}^1 \\ \boldsymbol{\xi}^2 \end{Bmatrix}$$

The generalised strains can still be written:

$$\mathbf{Z}^*(x, y) = \mathbf{B}^{+\mathrm{T}} \mathscr{L}(\Xi, x, y) \tag{79}$$

but now the complementary field solution $\mathbf{u}^*$ no longer needs to verify (77): the terms arising from the boundary $\partial_3 \boldsymbol{\Psi}$ no longer disappear. In particular, expression $\frac{1}{2}\langle \boldsymbol{\xi}, \mathbf{B}^{+\mathrm{T}} \mathscr{L}(\boldsymbol{\xi}) \rangle$ is no longer equal to the elastic energy of the 3-D field of stresses, and the pseudo-constitutive law (79), which is of the order of d, seems to us irrelevant.

On the other hand, although a tautology, it is very interesting to note that:

Remark

The polynomial solution provides an accurate assessment of the elastic energy in a finite element where the interpolation of the generalised strains is of degree d.

4.2.3. *Polynomial character of the solution with regard to* z *in a homogeneous milieu*

When the milieu is homogeneous and the constitutive law linear, with a tensor of the elasticity coefficients $\mathbf{R}$:

$$\mathscr{K}_3(\boldsymbol{\varepsilon}) = \mathbf{R} \cdot \boldsymbol{\varepsilon}$$

it is possible, as was said, to solve system (73) by polynomial identification. We seek λ_i polynomials of degree d in (x, y), and $\mathbf{u}$ as a polynome of degree d in (x, y) of which the coefficients are solutions of the linear differential equations with constant coefficients, obtained by identification from (73.1). For a full (and symmetrical) elasticity matrix this identification is not feasible manually, but can be done using formal calculation software, at least for small values of d.

Faced with a linear system the concern is, of course, that the solutions may be exponential but luckily not: all the eigenfrequencies are null.

Subsequently, the use of formal automatic calculations is not as straightforward as one may have hoped: the identification of the polynomes in (x, y, z) in system (73) produces more equations than unknowns. The software can then find those which are verified identically, than those which are linear combinations of others. This results in the right number of independent equations as soon as the degree with regard to z is at least three units higher than d, at least when d is less than or equal to two.

We are now going to present the outstanding features of the formal or numerical results obtained in this way in the case of a plate of uniform thickness.

4.3. *Elastic, homogeneous, isotropic plates, of uniform thickness*

4.3.1. *Approach to a constitutive law for an elastic, homogeneous, isotropic plate.*

We consider a field ξ of constant generalised strains. In order to save space notation $\mathbf{e}_{\text{mem}}$ is replaced by $\bar{\boldsymbol{\varepsilon}}$. The solution is thus as follows, E designating the Young modulus and v the Poisson coefficient.

Complementary displacement

$$u_1 = \gamma_1 z\left(\frac{1}{4}-\frac{5}{3}\frac{z^2}{h^2}\right); \quad u_2 = \gamma_2 z\left(\frac{1}{4}-\frac{5}{3}\frac{z^2}{h^2}\right)$$

$$u_3 = -\frac{v}{24(1-v)}[24z(\bar{\varepsilon}_{11}+\bar{\varepsilon}_{22})+(h^2-12z^2)(\chi_{11}+\chi_{22})] \tag{80}$$

3-D stresses

$$\sigma_{11} = \frac{E}{1-v^2}[\bar{\varepsilon}_{11}+z\chi_{11}+v(\bar{\varepsilon}_{22}+z\chi_{22})]; \quad \sigma_{22} = \frac{E}{1-v^2}[\bar{\varepsilon}_{22}+z\chi_{22}+v(\bar{\varepsilon}_{11}+z\chi_{11})]$$

$$\sigma_{12} = \frac{E}{2(1+v)}[\bar{\varepsilon}_{12}-z\chi_{12}]; \quad \sigma_{13} = \frac{5E}{2(1+v)}\left(\frac{1}{4}-\frac{z^2}{h^2}\right)\gamma_1; \quad \sigma_{23} = \frac{5E}{2(1+v)}\left(\frac{1}{4}-\frac{z^2}{h^2}\right)\gamma_2$$

Generalised stresses

$$N_{11} = \frac{Eh}{1-v^2}(\bar{\varepsilon}_{11}+v\bar{\varepsilon}_{22}); \quad N_{22} = \frac{Eh}{1-v^2}(\bar{\varepsilon}_{22}+v\bar{\varepsilon}_{11}); \quad N_{12} = \frac{Eh}{2(1+v)}\bar{\varepsilon}_{12}$$

$$T_1 = \frac{5}{6}\frac{Eh}{2(1+v)}\gamma_1 \quad T_2 = \frac{5}{6}\frac{Eh}{2(1+v)}\gamma_2$$

$$M_{11} = \frac{Eh^3}{12(1-v^2)}(\chi_{11}+v\chi_{22}); \quad M_{11} = \frac{Eh^3}{12(1-v^2)}(\chi_{22}+v\chi_{11}); \quad M_{12} = \frac{Eh^3}{24(1+v)}\chi_{12}$$

"Correction coefficient" of shearing: $\frac{5}{6}$.

This solution appears in all the textbooks on Strength of Materials, but is classically obtained by using the equilibrium equation which links the shear effort to the divergence at the bending moment: this implies a linear bending moment M_{ii} in (x, y) when shear force T_i is constant, and consequently also a linear curvature χ_{ii}; this situation will be examined in sub-paragraph 4.3.3. Here, we consider the generalised strains to be constant.

Let us illustrate this solution by three graphs, plotted when all the generalised strains are null except:

$$\gamma_1 = 1$$

with:

$$E = 1; \quad v = 0.25; \quad h = 1$$

The total displacement is calculated by attributing 0 to the values at the origin of principal displacement **V** and quantities $(\partial v_1/\partial y)-(\partial v_2/\partial x)$, $\partial w/\partial x$ and $\partial w/\partial y$.

4.3.2. *Polynomial solution for field* ξ *of degree* 2.
The generalised strain is written:

$$\xi = \xi^0+\xi^1 x+\xi^2 y+\xi^3 x^2+\xi^4 xy+\xi^5 y^2 = \Xi^{\mathrm{T}}\begin{Bmatrix}1\\x\\y\\x^2\\xy\\y^2\end{Bmatrix} \quad \text{with:}\quad \boldsymbol{\Xi}\begin{Bmatrix}\xi^0\\ \xi^1\\ \xi^2\\ \xi^3\\ \xi^4\\ \xi^5\end{Bmatrix} \tag{81}$$

and:

$$\xi^i = (\bar{\varepsilon}^i_{11}, \bar{\varepsilon}^i_{22}, \bar{\varepsilon}^i_{12}, \gamma^i_1, \gamma^i_2, \chi^i_{11}, \chi^i_{22}, \chi^i_{12}) \quad i = 0, 1, \ldots, 5$$

In order to save space, the complementary displacement and the force field are given at $x = y = 0$, and terms of the same order are grouped together. This obviously makes it possible to complete the solution by changing variables.

Completely formal calculation being too time-consuming, value 0.25 was chosen for Poisson coefficient ν.

Complementary displacement
Terms of the order of 0

$$u^0_1 = 0.25z\left(1-1.67\frac{z^2}{h^2}\right)\gamma^0_1;\quad u^0_2 = 0.25z\left(1-1.67\frac{z^2}{h^2}\right)\gamma^0_2$$

$$u^0_3 = -0.333(\bar{\varepsilon}^0_{11}+\bar{\varepsilon}^0_{22})-(0.0139h^2-0.167z^2)(\chi^0_{11}+\chi^0_{22}) \tag{82}$$

Terms of the order of 1

$$u^1_1 = -(0.0139h^2-0.167z^2)(\bar{\varepsilon}^1_{11}+\bar{\varepsilon}^1_{22})+z(0.00694h^2-0.0463z^2)(\chi^1_{11}+\chi^1_{22})$$

$$u^1_2 = -(0.0139h^2-0.167z^2)(\bar{\varepsilon}^2_{11}+\bar{\varepsilon}^2_{22})+z(0.00694h^2-0.0463z^2)(\chi^2_{11}+\chi^1_{22})$$

$$u^1_3 = \left(0.00579h^2-0.111z^2+0.278\frac{z^4}{h^2}\right)(\gamma^1_1+\gamma^2_2)$$

Terms of the order of 2

$$u^2_1 = z\left(0.0101h^2-0.117z^2+0.278\frac{z^4}{h^2}\right)\gamma^3_1+z\left(0.00744h^2-0.0794z^2+0.167\frac{z^4}{h^2}\right)\gamma^5_1$$

$$+z\left(0.00132h^2-0.0187z^2+0.0556\frac{z^4}{h^2}\right)\gamma_2^4$$

$$u_2^2 = z\left(0.0101h^2-0.117z^2+0.278\frac{z^4}{h^2}\right)\gamma_2^5+z\left(0.00744h^2-0.0794z^2+0.167\frac{z^4}{h^2}\right)\gamma_2^3$$

$$u_3^2 = z(0.00926h^2-0.037z^2)(\bar{\varepsilon}_{11}^3+\bar{\varepsilon}_{11}^5+\bar{\varepsilon}_{22}^3+\bar{\varepsilon}_{22}^5)$$
$$+(0.0000514h^4-0.00154h^2z^2+0.00617z^4)(\chi_{11}^3+\chi_{11}^5+\chi_{22}^3+\chi_{22}^5)$$

Stresses (for a Young modulus equal to 1).

Terms of the order of 0

$$\sigma_{11}^0 = 1.07\bar{\varepsilon}_{11}^0+0.267\bar{\varepsilon}_{22}^0-z(1.07\chi_{11}^0+0.267\chi_{22}^0)$$

$$\sigma_{22}^0 = 0.267\bar{\varepsilon}_{11}^0+1.07\bar{\varepsilon}_{22}^0-z(0.267\chi_{11}^0+1.07\chi_{22}^0)$$

$$\sigma_{33}^0 = 0$$

$$\sigma_{12}^0 = 0.4\bar{\varepsilon}_{12}^0-0.4z\chi_{12}^0$$

$$\sigma_{13}^0 = \left(0.5-2\frac{z^2}{h^2}\right)\gamma_1^0 \quad \sigma_{23}^0 = \left(0.5-2\frac{z^2}{h^2}\right)\gamma_2^0 \tag{83}$$

Terms of the order of 1

$$\sigma_{11}^1 = \left(0.211z-1.56\frac{z^3}{h^3}\right)\gamma_1^1+\left(0.0111z-0.222\frac{z^3}{h^3}\right)\gamma_2^2$$

$$\sigma_{22}^1 = \left(0.0111z-0.222\frac{z^3}{h^3}\right)\gamma_1^1+\left(0.211z-1.56\frac{z^3}{h^3}\right)\gamma_2^2$$

$$\sigma_{33}^1 = \left(-0.167z+0.667\frac{z^3}{h^3}\right)(\gamma_1^1+\gamma_2^2)$$

$$\sigma_{12}^1 = \left(0.1z-0.667\frac{z^3}{h^3}\right)(\gamma_1^2+\gamma_2^1)$$

$$\sigma_{13}^1 = (-0.00278h^2+0.0111z^2)(\chi_{11}^1+\chi_{22}^1); \quad \sigma_{23}^1 = (-0.00278h^2+0.0111z^2)(\chi_{11}^2+\chi_{22}^2)$$

Terms of the order of 2

$$\sigma_{11}^2 = (-0.0296h^2+0.356z^2)(\bar{\varepsilon}_{11}^3+\bar{\varepsilon}_{22}^3)+(-0.00741h^2+0.0889z^2)(\bar{\varepsilon}_{11}^5+\bar{\varepsilon}_{22}^5)$$
$$+z(0.0154h^2-0.101z^2)(\chi_{11}^3+\chi_{22}^3)+z(-0.00432h^2+0.0272z^2)(\chi_{11}^5+\chi_{22}^5)$$

$$\sigma_{22}^2 = (-0.00741h^2+0.0889z^2)(\bar{\varepsilon}_{11}^3+\bar{\varepsilon}_{22}^3)+(-0.0296h^2+0.356z^2)(\bar{\varepsilon}_{11}^5+\bar{\varepsilon}_{22}^5)$$

$$+z(-0.00432h^2+0.0272z^2)(\chi_{11}^3+\chi_{22}^3)+z(0.0154zh^2-0.101z^2)(\chi_{11}^5+\chi_{22}^5)$$

$$\sigma_{33}^2 = z(0.00185h^2-0.0741z^2)(\chi_{11}^3+\chi_{22}^3+\chi_{11}^5+\chi_{22}^5)$$

$$\sigma_{12}^2 = (-0.0111h^2+0.133z^2)(\bar{\varepsilon}_{11}^4+\bar{\varepsilon}_{22}^4)+z(0.00556h^2-0.037z^2)(\chi_{11}^4+\chi_{22}^4)$$

$$\sigma_{13}^2 = \left(0.00866h^2-0.229z^2+0.778\frac{z^4}{h^2}\right)\gamma_1^3+\left(0.00298h^2-0.0952z^2+0.333\frac{z^4}{h^2}\right)\gamma_1^5$$

$$+\left(0.0284h^2-0.0669z^2+0.222\frac{z^4}{h^2}\right)\gamma_2^4$$

$$\sigma_{23}^2 = \left(0.0284h^2-0.0669z^2+0.222\frac{z^4}{h^2}\right)\gamma_1^4+\left(0.00298h^2-0.0952z^2+0.333\frac{z^4}{h^2}\right)\gamma_2^3$$

$$+\left(0.00866h^2-0.229z^2+0.778\frac{z^4}{h^2}\right)\gamma_2^5$$

4.3.3. *Solution of Saint-Venant's type*

The above solution is now used for null membrane strains $\bar{\boldsymbol{\varepsilon}}$. We consider a region of the plate where the external efforts are null ($p = 0$, $\mathbf{c} = \mathbf{0}$). The equation set to be solved by formal computation is now:

$$\bar{\boldsymbol{\varepsilon}} = 0 \quad (84.1)$$

$$\mathrm{div}^{(2)}\,\mathbf{T} = 0 \quad (84.2)$$

$$\mathbf{div}^{(2)}\,\mathbf{M}+\mathbf{T} = \mathbf{0} \quad \text{Compatibility equations (33) and (34)} \quad (84.3)$$

First it comes that eqn (84.2) implies that σ_{33} vanishes throughout the thickness, which confirms the main hypothesis of Kirchhoff and Reissner's theories.

Finally a general polynomial solution is exhibited. For sake of space we give only the relations concerned by displacements, and also the values of the generalised stress component T_1, together with 3-D-stress σ_{13}. Integration constants w_0, b_1 and b_2 are arbitrary.

Principal displacement

$$V_1 = b_1z+1.91\frac{x^2z}{h^2}\gamma_1^0+0.638\frac{x^3z}{h^2}\gamma_1^1+yz\left(0.5+3.09\frac{x^2}{h^2}-0.636\frac{y^2}{h^2}\right)\gamma_1^2$$

$$-yz\left(0.5+1.91\frac{x^2}{h^2}-1.307\frac{y^2}{h^2}\right)\gamma_2^1-z(x\chi_{11}^0+y\chi_{12}^0+xy\chi_{11}^2-0.309x^2\chi_{22}^1+0.5y^2\chi_{22}^1)$$

$$V_2 = b_2z+1.91\frac{y^2z}{h^2}\gamma_2^0-0.638\frac{y^3z}{h^2}\gamma_1^1-xz\left(0.5-1.031\frac{x^2}{h^2}+1.91\frac{y^2}{h^2}\right)\gamma_1^2$$

$$+xz\left(0.5-0.636\frac{x^2}{h^2}+3.09\frac{y^2}{h^2}\right)\gamma_2^1-z(y\chi_{22}^0+x\chi_{12}^0+xy\chi_{22}^1+0.5x^2\chi_{11}^2-0.309y^2\chi_{11}^2)$$

$$V_3=w_0-b_1x-b_2y+x\left(1-0.638\frac{x^2}{h^2}\right)\gamma_1^0+y\left(1-0.638\frac{y^2}{h^2}\right)\gamma_2^0$$

$$+x^2\left(0.5-0.16\frac{x^2}{h^2}\right)\gamma_1^1-y^2\left(0.5-0.16\frac{y^2}{h^2}\right)\gamma_1^1$$

$$+xy\left(0.5-1.03\frac{x^2}{h^2}+0.636\frac{y^2}{h^2}\right)\gamma_1^2+xy\left(0.5+0.636\frac{x^2}{h^2}-1.03\frac{y^2}{h^2}\right)\gamma_2^1$$

$$+x^2(0.5\chi_{11}^0+0.5y\chi_{11}^2-0.103x\chi_{22}^1)+xy\chi_{12}^0+y^2(0.5\chi_{22}^0+0.5x\chi_{22}^1-0.103y\chi_{11}^2)$$

Complementary displacement

$$u_1=z\left(0.223-1.49\frac{z^2}{h^2}\right)(\gamma_1^0+z\gamma_1^1)+yz\left(0.234-1.56\frac{z^2}{h^2}\right)\gamma_1^2-yz\left(0.0164-0.11\frac{z^2}{h^2}\right)\gamma_2^1$$

$$+z(0.00266h^2-0.0177z^2)\chi_{22}^1$$

$$u_2=z\left(0.223-1.49\frac{z^2}{h^2}\right)(\gamma_2^0-z\gamma_1^1)-xz\left(0.0164-0.11\frac{z^2}{h^2}\right)\gamma_1^2+xz\left(0.234-1.56\frac{z^2}{h^2}\right)\gamma_2^1$$

$$+z(0.00266h^2-0.0177z^2)\chi_{11}^2$$

$$u_3=\left(0.0532-0.638\frac{z^2}{h^2}\right)(x\gamma_1^0+y\gamma_{12}^0)+\left(0.266-0.319\frac{z^2}{h^2}\right)(x^2-y^2)\gamma_1^1$$

$$+xy\left(0.0329-0.395\frac{z^2}{h^2}\right)(\gamma_1^2+\gamma_2^1)$$

$$-(0.0139h^2-0.167z^2)(\chi_{11}^0+\chi_{22}^0)-(0.00532h^2-0.638z^2)(y\chi_{11}^2+x\chi_{22}^1)$$

Remark

A slight dissymmetry appears in the above formulas; indeed, compatibility equations imply

$$\gamma_1^1+\gamma_2^2=0$$

which need the elimination of any one of these quantities, here γ_2^2.

Shear stresses

$$\sigma_{13}=\left(0.511-2.04\frac{z^2}{h^2}\right)\gamma_1^0+\left(0.511-2.04\frac{z^2}{h^2}\right)x\gamma_1^1+\left(0.507-2.03\frac{z^2}{h^2}\right)y\gamma_1^2$$

$$\left(0.00658-0.0263\frac{z^2}{h^2}\right)y\gamma_2^1-(0.00106h^2-0.00426z^2)\chi_{22}^1$$

$$T_1 = 0.34h(\gamma_1^0+x\gamma_1^1)+0.338y\gamma_1^2+0.00439hy\gamma_2^1-0.000709\chi_{22}^1$$

The transverse shear correction coefficient derived from the final formula is 0.851, as far as the curvature term χ_{22}^1 is neglected. In fact this coefficient depends slightly on the choice of eliminated variables, which was expected, since there is no exact local plate constitutive equation. Another set of eliminated variables gives the usual 0.833, and a less symmetrical solution than above.

4.4. *Multi-layer plates*

4.4.1. *System to be solved for a multi-layer plate*

When the plate, of uniform[3] thickness h, is made up of N homogeneous layers, the following system must be solved. It derives from (73), where continuity conditions at the border between two successive layers have been introduced:

Differential equation

$$\mathbf{div}\,[\mathbf{R}^k\cdot(\mathbf{B}^+\xi+\mathbf{grad}_s\mathbf{u}^k)] = (\lambda_1+\lambda_4 z)\mathbf{i}+(\lambda_2+\lambda_5 z)\mathbf{j}+\lambda_3\mathbf{k} \tag{85.1}$$

Continuity of stresses and displacements

$$u_i^{k+1}(x,y,z_{k+1}) = u_{i+1}^{k+1}(x,y,z_{k+1}) \tag{85.2}$$

for $k = 1,\ldots,N-1$ and $i = 1,2,3$:

$$\sigma_{i3}^{k+1}(x,y,z_{k+1}) = \sigma_{i3}^{k}(x,y,z_{k+1}) \tag{85.3}$$

Upper and lower faces,

$$\sigma_{i3}^1\left(x,y,-\frac{h}{2}\right) = 0 \tag{85.4}$$

not loaded, for $i = 1,2,3$:

$$\sigma_{i3}^N\left(x,y,\frac{h}{2}\right) = 0 \tag{85.5}$$

Constraint equations

$$\sum_{k=1,\ldots,N}\int_{z_k}^{z_{k+1}}\mathbf{u}^k\,\mathrm{d}z = 0\ \sum_{k=1,\ldots,N}\int_{z_k}^{z_{k+1}}u_1^k z\,\mathrm{d}z = 0;\quad \sum_{k=1,\ldots,N}\int_{z_k}^{z_{k+1}}u_2^k z\,\mathrm{d}z = 0 \tag{85.6}$$

Side z_k is that of the lower face of layer no. k:

[3] And probably also when h is a polynomial of degree n.

$$z_1 = -\frac{h}{2}; \quad \ldots; \quad z_{N+1} = \frac{h}{2}$$

In order to be able to compare our results with those given in the book written by Batoz and Dhatt (1990), we chose to consider a plate composed of three layers of the same orthotropic material, the layer in the middle being twice as thick as the others, oriented perpendicularly. The interfaces are thus:

$$z_1 = -\frac{h}{2}; \quad z_2 = -\frac{h}{4}; \quad z_3 = \frac{h}{4}; \quad z_4 = \frac{h}{2}$$

To avoid manipulating four indices we consider that the stiffness matrix transforms the strain column:

$$\langle \varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{13}, 2\varepsilon_{23} \rangle$$

into the force column:

$$\langle \sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23} \rangle$$

The stiffness matrices $[\mathbf{R}^k]$ are, respectively for layers 1 and 3 on the one hand, and for layer 2 on the other:

$$[\mathbf{R}^1] = [\mathbf{R}^3] = \begin{bmatrix} 25.2 & 0.336 & 0.336 & 0 & 0 & 0 \\ 0.336 & 1.07 & 0.271 & 0 & 0 & 0 \\ 0.336 & 0.271 & 1.07 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2 \end{bmatrix};$$

$$[\mathbf{R}^2] = \begin{bmatrix} 1.07 & 0.336 & 0.271 & 0 & 0 & 0 \\ 0.336 & 25.2 & 0.336 & 0 & 0 & 0 \\ 0.271 & 0.336 & 1.07 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}$$

It is, thus, a composite material, the fibres of which are oriented along the x axis for the odd layers.

It is also possible to calculate the polynomial solution when field ξ is a polynomial of degree 2 in (x, y). As several pages would be necessary to write it in full, we shall limit ourselves here to giving the approximate constitutive equation, itself extremely informative.

4.4.2. *Multi-layer plate constitutive law*

We consider a constant field ξ of generalised strains and the following solution is found for the numerical data given above. The upperscript in the brackets indicates the layer number.

Complementary displacement

$$u_1^{(1)} = -\left(0.196h+0.429z+0.761\frac{z^3}{h^2}\right)\gamma_1$$

$$u_1^{(2)} = \left(0.427z-1.9\frac{z^3}{h^2}\right)\gamma_1$$

$$u_1^{(3)} = \left(0.196h-0.429z-0.761\frac{z^3}{h^2}\right)\gamma_1$$

$$u_2^{(1)} = \left(0.328h+1.38z-3.18\frac{z^3}{h^2}\right)\gamma_2$$

$$u_2^{(2)} = -\left(0.0462z+1.27\frac{z^3}{h^2}\right)\gamma_2$$

$$u_2^{(3)} = \left(-0.328h+1.38z-3.18\frac{z^3}{h^2}\right)\gamma_2$$

$$u_3^{(1)} = -(0.015h+0.313z)\bar{\varepsilon}_{11}+(0.015h-0.253z)\bar{\varepsilon}_{22} + (-0.0137h^2+0.157z^2)\chi_{11}+(-0.00992h^2+0.127z^2)\chi_{22}$$

$$u_3^{(2)} = -0.253z\bar{\varepsilon}_{11}-0.313z\bar{\varepsilon}_{22} + (-0.0118h^2+0.127z^2)\chi_{11}+(-0.00118h^2+0.157z^2)\chi_{22}$$

$$u_3^{(3)} = (0.015h-0.313z)\bar{\varepsilon}_{11}-(0.015h+0.253z)\bar{\varepsilon}_{22} + (-0.0137h^2+0.157z^2)\chi_{11}+(-0.00992h^2+0.127z^2)\chi_{22}$$

3-*D*-*Stresses*

$$\sigma_{11}^{(1)} = 25.1\bar{\varepsilon}_{11}+0.251\bar{\varepsilon}_{22}-25.1z\chi_{11}-0.251z\chi_{22}$$

$$\sigma_{11}^{(2)} = 1.00\bar{\varepsilon}_{11}+0.251\bar{\varepsilon}_{22}-1.00z\chi_{11}-25.1z\chi_{22}$$

$$\sigma_{11}^{(3)} = 25.1\bar{\varepsilon}_{11}+0.251\bar{\varepsilon}_{22}-25.1z\chi_{11}-0.251z\chi_{22}$$

$$\sigma_{22}^{(1)} = 0.251\bar{\varepsilon}_{11}+1.00\bar{\varepsilon}_{22}-0.251z\chi_{11}-1.00z\chi_{22}$$

$$\sigma_{22}^{(2)} = 0.251\bar{\varepsilon}_{11}+25.1\bar{\varepsilon}_{22}-0.251z\chi_{11}-25.1z\chi_{22}$$

$$\sigma_{22}^{(3)} = 0.251\bar{\varepsilon}_{11}+1.00\bar{\varepsilon}_{22}-0.251z\chi_{11}-1.00z\chi_{22}$$

$$\sigma_{33}^{(1)} = 0; \quad \sigma_{33}^{(2)} = 0; \quad \sigma_{33}^{(3)} = 0$$

$$\sigma_{12}^{(1)} = \sigma_{12}^{(2)} = \sigma_{12}^{(3)} = 0.5\bar{\varepsilon}_{12}-0.5z\chi_{12}$$

$$\sigma_{13}^{(1)} = \sigma_{13}^{(2)} = \sigma_{13}^{(3)} = \left(0.285-1.14\frac{z^2}{h^2}\right)\gamma_1$$

$$\sigma_{23}^{(1)} = \sigma_{23}^{(2)} = \sigma_{23}^{(3)} = \left(0.477 - 1.91\frac{z^2}{h^2}\right)\gamma_2$$

Generalised stresses

$$N_{11} = 13.0h\bar{\varepsilon}_{11} + 0.251h\bar{\varepsilon}_{22}; \quad N_{22} = 13.0h\bar{\varepsilon}_{22} + 0.251h\bar{\varepsilon}_{11}; \quad N_{12} = 0.5h\bar{\varepsilon}_{12}$$

$$T_1 = 0.190h\gamma_1 \quad T_2 = 0.318h\gamma_2$$

$$M_{11} = 1.84h^3\gamma_{11} + 0.0209h^3\gamma_{22}; \quad M_{22} = 0.0209h^3\gamma_{11} + 0.334h^3\gamma_{22}; \quad M_{12} = 0.0417h^3\gamma_{12}$$

Shear correction factors: 0.544 in x, 0.908 in y, to be compared with values 0.57 and 0.88 given by Batoz and Dhatt (1990), p. 250.

We notice that each of the shear stresses σ_{12}, σ_{13} and σ_{23} is the same polynomial, of the first or second degree, throughout the thickness; Batoz and Dhatt give, on the contrary, shear forces σ_{13} and σ_{23} which have discontinuous z-derivatives (cf Batoz and Dhatt (1990), p. 249). On the other hand, we find, like them, traction stresses σ_{11} and σ_{22} which have discontinuous z-derivatives.

The graphs below are to be compared with Figs 3, 4 and 5 illustrating the isotropic homogeneous plate. They show the smooth (parabolic) curve of the shear stress, in contrast with the jagged graph of the complementary displacement. The total displacement is of zero derivative on the upper and lower faces.

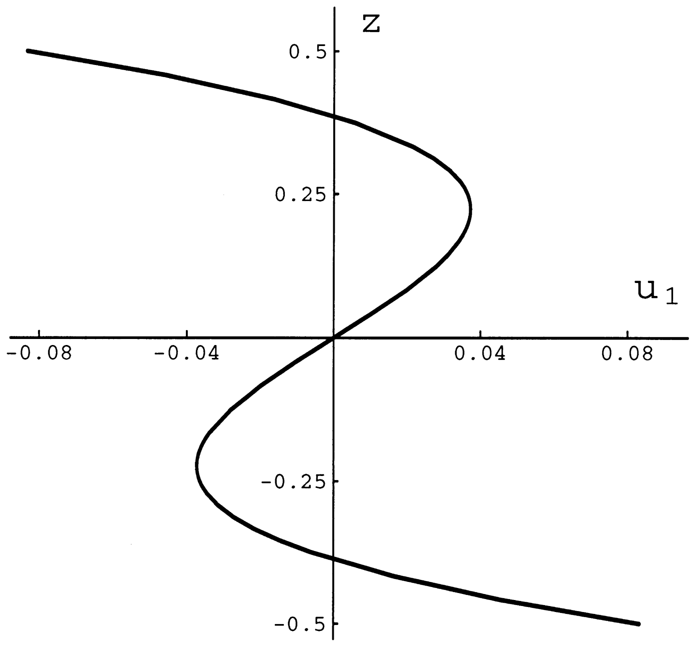

Fig. 3. Complementary displacement.

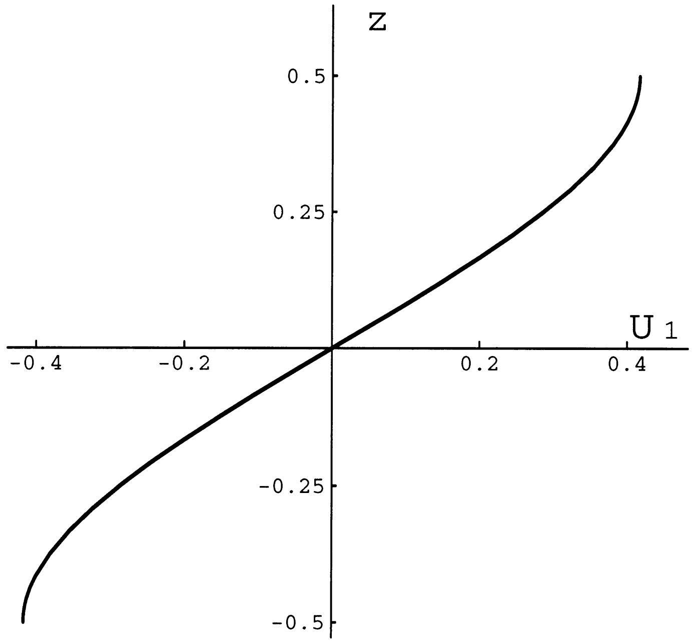

Fig. 4. Total displacement.

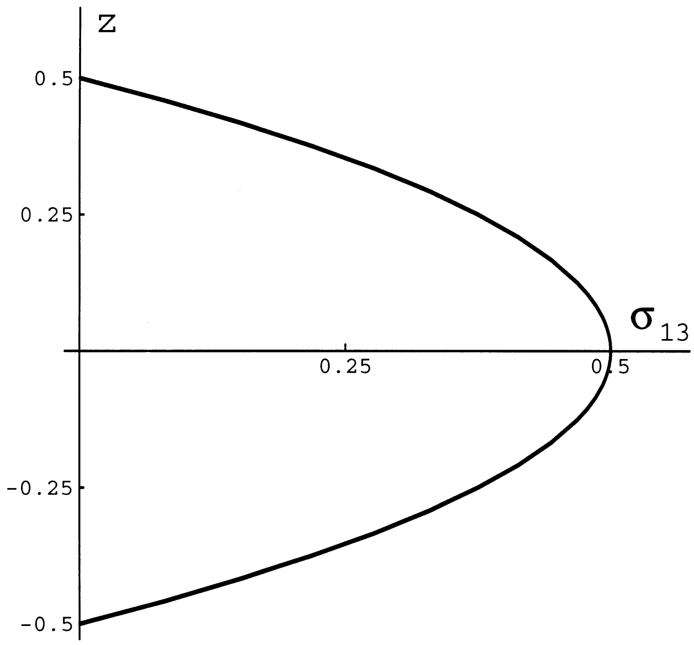

Fig. 5. Shear stress.

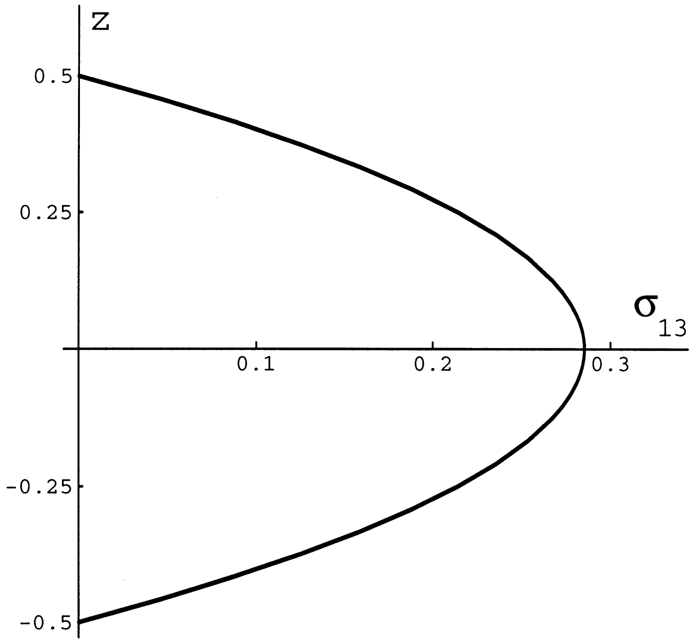

Fig. 6. Stress.

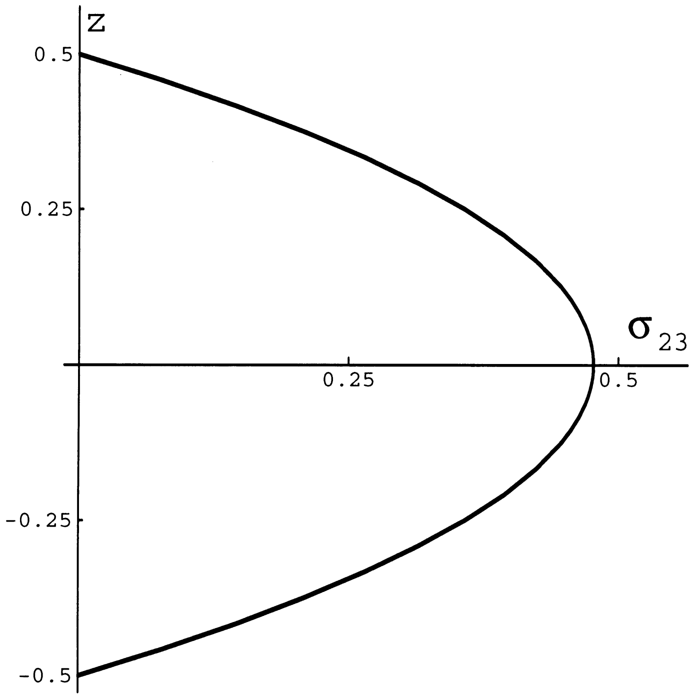

Fig. 7. Stress.

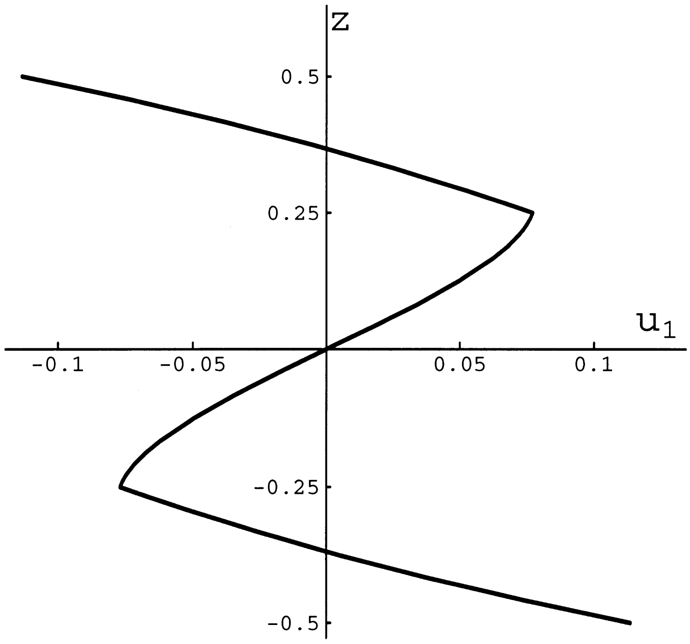

Fig. 8. Complementary displacement u_1.

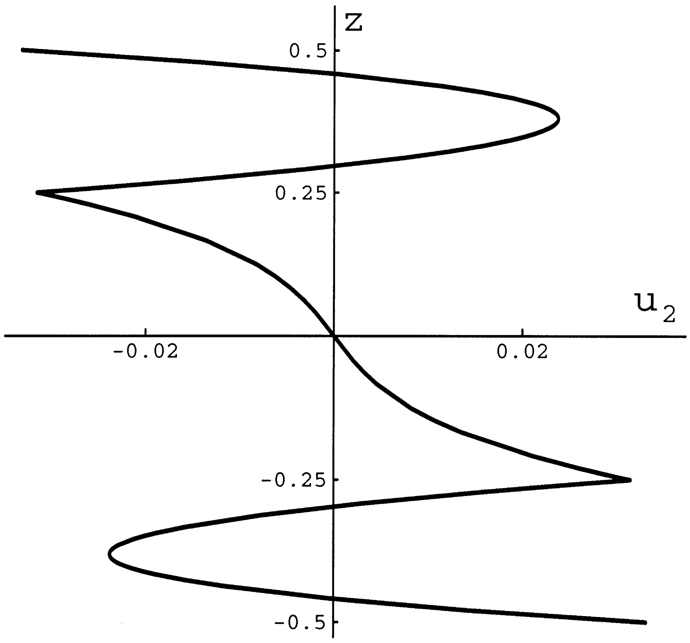

Fig. 9. Complementary displacement u_2.

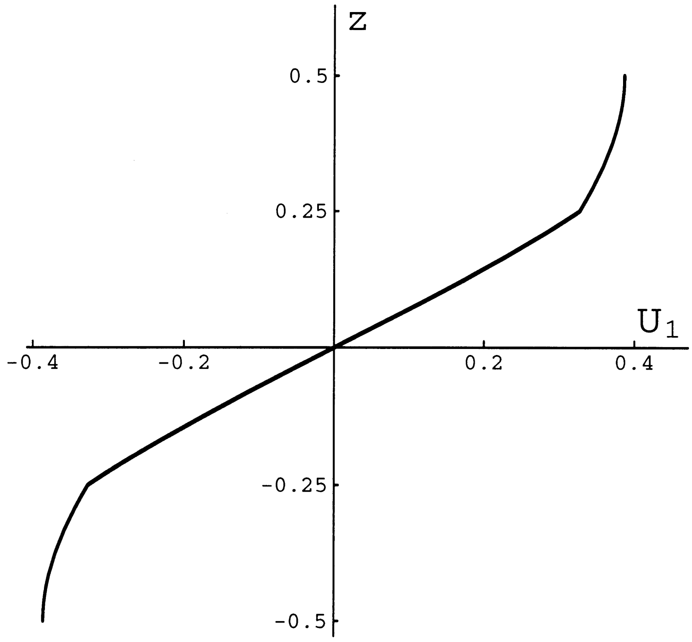

Fig. 10. Total displacement U_1.

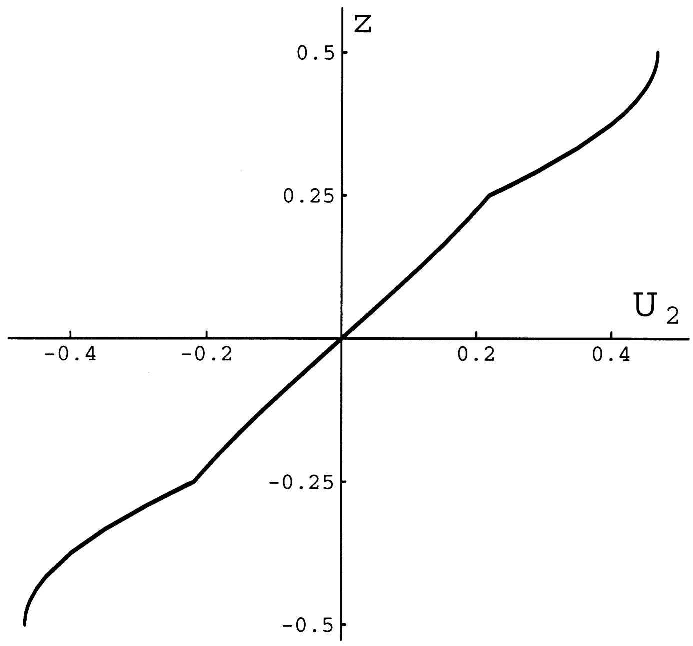

Fig. 11. Total displacement U_2.

5. Conclusions

A completely new plate theory has been developed, free from the usual hypotheses concerning normals, such as the Kirchhoff or Reissner–Mindlin hypotheses, or regarding stresses, such as the plane stress hypothesis. Moreover, all the theoretical articles which the author had the chance to read deal with an elastic 3-D-behaviour; on the contrary our theory does not depend on the 3-D-constitutive equation.

The clue to this approach lies in introducing kinematics which split the total 3-D-displacement into: a principal displacement, which verifies the Reissner–Mindlin hypothesis and establishes the algebraic links with the usual plate theory, and a complementary displacement which preserves the exactness of the equilibrium equations and enables the constitutive equation of the plate to be derived from the tridimensional one.

Notice that the principal displacement kinematics could have been chosen simpler, verifying the Kirchhoff hypothesis for instance. That would reduce the space of principal displacements, enlarge that of complementary displacements, and need some slight changes in the present theory. In particular, the latter is in no way a so-called "high order" theory, but, on the contrary, can be easily adapted to any kinematics chosen for the principal displacements.

In general, of course, the tridimensional equilibrium problem does not reduce to a plate equilibrium problem. A condition, named "property $\mathscr{P}$" is needed for the external loads: one must be able to express their constitutive equations in terms of the principal displacement only, and their virtual work in terms of the virtual principal displacement only. This condition may be summed up by saying that external loads are just plate loads. Thus, property $\mathscr{P}$ is nothing but a necessary condition for the existence of any plate theory! In the classical theories property $\mathscr{P}$ is involved in the Kirchhoff or Reissner–Mindlin hypotheses.

In order to ensure the existence of a local plate constitutive equation we have had to make a second hypothesis: the plate is thin enough. This hypothesis is necessary in any plate theory since, by definition, a local plate constitutive law cannot take into account the possible proximity of an edge. Finally, it appears that our theory is free from any unnecessary hypothesis.

Comparison have been given with some classical results in the case of an elastic behaviour. The results are identical in the case of an isotropic and homogeneous material, somewhat different when a multi-layer plate is considered, even if the correction shear factor is approximately the same. By using formal computations exact solutions have been exhibited for polynomial data, which provides a new basis for building Saint-Venant solutions as well as exact elastic plate finite elements.

Further worthwhile developments would be concerned with asymptotic plasticity: limit loads and shake-down domains.

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